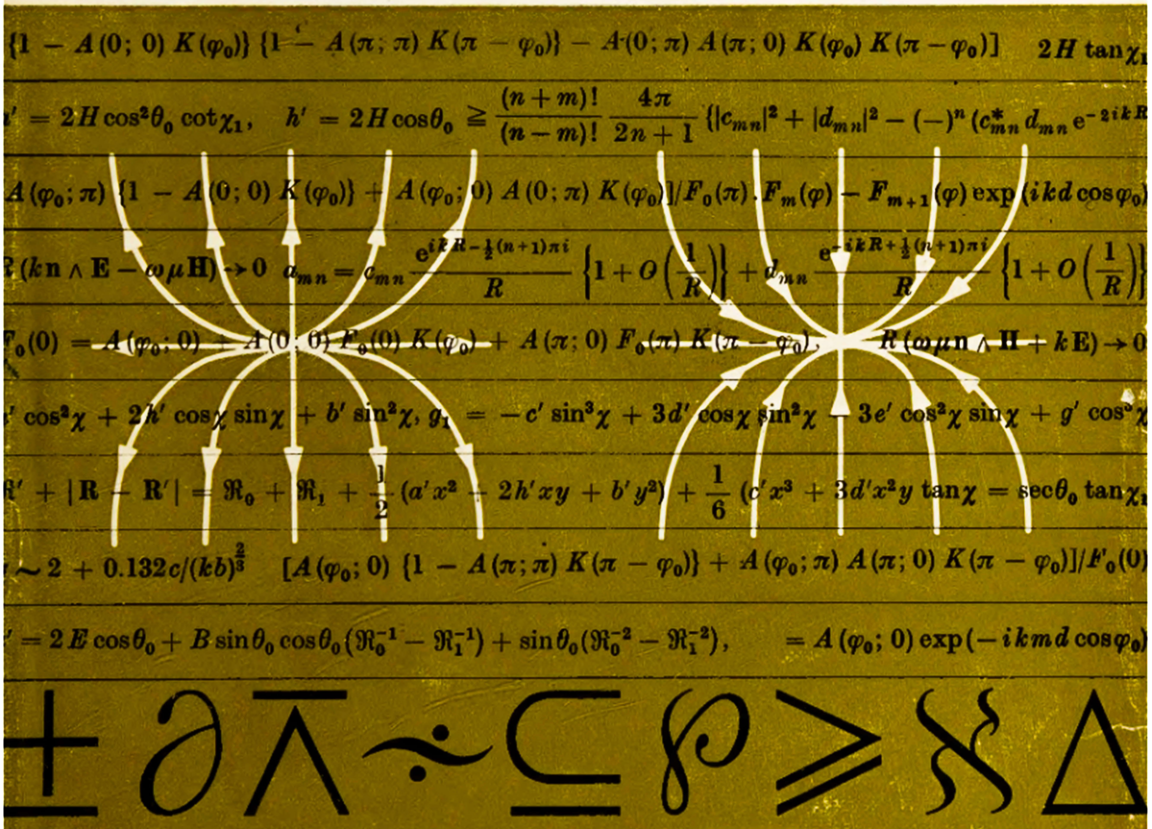
Functions of a Complex Variable and some of their applications

VOLUME II

B. A. Fuchs and V. I. Levin



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FOREWORD

THE present book is intended to form a supplement to Vol. I of "Functions of a Complex Variable and Some of Their Applications" by B. A. Fuchs and B. V. Shabat.†

It is intended for engineers, and also for students and postgraduate students of the higher technical colleges, who wish to make themselves more familiar with some special questions in the theory of the functions of a complex variable and its applications to those chapters of mathematical analysis which play an important part in the solution of technical and physical problems (such as differential equations, the operational calculus, special functions and questions of stability).

The present book, together with the above-mentioned book by B. A. Fuchs and B. V. Shabat, will introduce the reader to the principles of the theory of the functions of a complex variable. It will acquaint him, although, of course, not exhaustively, with the special sections of this theory, which from the point of view of applications are the most important, and will give him an idea how to use its methods for the solution of applied problems.

The reader is therefore assumed to have a knowledge of the foundations of complex analysis, which can be acquired from the book by B. A. Fuchs and B. V. Shabat mentioned above. Since the present book contains a large number of references to it, it will be referred to simply as F.C.V.

For Chapters I, II and V of the present book it is sufficient to know the bases of the theory of the functions of a complex variable as set out in the version of the abbreviated study of F.C.V., which, as is stated in the Foreword to that book, gives an introduction to the elementary theory of functions. For Chaps. III and IV the second version of the abbreviated study of F.C.V., which gives the necessary introduction to operational calculus, is sufficient. The knowledge necessary can also be gained from other books containing the foundations of the general theory of the functions of a complex variable.

† Originally published in Russian in 1949. English edition published by Pergamon Press Ltd., 1961.

The book deals with the following questions: the analytic theory of differential equations (Chaps. I and II); the Laplace transformation and its applications (Chaps. III and IV) and Hurwitz's problem for polynomials (Chap. V).

Chapter I is of a preliminary character and is devoted to algebraic functions. Principal attention is given to the study of these functions in the neighbourhoods of their regular and singular points and to their expansions into series of powers of the variable (integral and fractional, positive and negative).

Chapter II begins with a short section dealing with the analytic functions $f(w, z)$ of two complex variables. Then differential equations of the form $dw/dz = f(w, z)$ are considered for cases where the function $f(w, z)$ is regular in the neighbourhood of the initial values w_0, z_0 of the variables w, z, where it has at $w = w_0$, $z = z_0$ a pole or, lastly, a point of indeterminateness. Linear differential equations of the second order are then considered, and the results obtained are applied to the Euler–Bessel equation and its integrals (cylinder functions).

Chapter III sets out the fundamental properties of the Laplace transformation and the principles of its application to the study of special functions and to the integration of differential equations. It must be emphasized that this material is treated here as a chapter on the theory of the functions of a complex variable. The authors have sought to deal simply with the most important facts on the subject, but at the same time more precisely and more strictly than is usually done in handbooks on the operational calculus for engineers. It has not been the authors' aim to set out the actual apparatus of the operational calculus, since this can be found in many books.

Chapter IV is devoted to contour integration and asymptotic expansions. As is well known, these are of great practical value and have in general been inadequately dealt with in other books on the subject. In addition, an account is given here of the fundamental facts of the theory of asymptotic expansions, and examples of these expansions are analysed in detail. It should be noted that for the study of these questions, some knowledge is necessary of the theory of many-valued and, in particular, algebraic functions. This is to be found in the first chapter.

Chapter V treats Hurwitz's problem for polynomials. Although it is not connected with the preceding material, the authors are agreed that its importance for a large number of applications makes its

consideration essential in a book dealing with the special problems of the theory of functions of a complex variable.

In each chapter the reader will find a large number of worked examples. These are intended to serve not only as illustrations of the theoretical conclusions, but also as models for solving independently any similar problems encountered.

The engineer does not usually meet the questions dealt with in this book until his problems have been already formulated mathematically. The authors have not therefore found it necessary to consider problems with marked physical content. Nevertheless, in a number of places they have emphasized the links that exist between the mathematical theories treated and the corresponding technical disciplines.

Chapters I and II were written by B. A. Fuchs, Chapters III and IV by V. I. Levin, and Chapter V by both authors together. However, the close contact constantly maintained between them during their work on the book makes them equally responsible for the volume as a whole.

CONTENTS

V. HURWITZ'S PROBLEM FOR POLYNOMIALS

CHAPTER I

ALGEBRAIC FUNCTIONS

In the present chapter algebraic functions will be considered. They are defined as solutions of equations of the form

$$\Phi(w, z) = 0, \tag{1.1}$$

where $\Phi(w, z)$ is a polynomial in the variables w and z. We shall obtain an algebraic function if we express, from equation (1.1), one of the variables w or z in terms of the other.

1. Polynomials in two variables

Let us consider the polynomial $\Phi(w, z)$ in two complex variables w and z, that is an expression of the form

$$\Phi(w, z) = \phi_0(z)w^n + \phi_1(z)w^{n-1} + \ldots + \phi_n(z). \tag{1.2}$$

Here $\phi_j(z), j = 0, 1, \ldots n$, are polynomials in z of degree, not exceeding a certain number m, where the degree of at least one of the polynomials $\phi_j(z)$ is actually equal to m. In addition to this, it is supposed that $\phi_0(z) \not\equiv 0$. Hence $\Phi(w, z)$ is a polynomial of degree n with respect to w and of degree m with respect to z. It can also be expanded in powers of z; it is possible to represent it as the sum of homogeneous polynomials written in the order of their descending powers.

The latter representation is obtained as follows: the polynomial $\Phi(w, z)$ is a finite sum of monomials of the form $a_{pq}w^pz^q$, where the coefficients a_{pq} are complex constants. The sum $p+q = k$ is called the degree of the monomial $a_{pq}w^pz^q$. If we collect all the monomials with the same degree, occurring in our polynomial $\Phi(w, z)$ into homogeneous polynomials $\Phi^{(k)}(w, z)$ (k is the degree of the monomials occurring in this polynomial) we obtain the desired decomposition in the form

$$\Phi(w, z) = \Phi^{(0)}(w, z) + \Phi^{(1)}(w, z) + \ldots + \Phi^{(l)}(w, z) \tag{1.3}$$

The number l, the greatest value of the quantity $p+q$ for the monomials, entering into the composition of the polynomial $\Phi(w, z)$, is

called its *degree.* We shall suppose that every homogeneous polynomial $\Phi^{(k)}(w, z)$ is arranged in descending powers of the variable w and has the form

$$\Phi^{(k)}(w, z) = a_{k0}w^k + a_{k1}w^{k-1}z + \dots + a_{0k}z^k. \tag{1.4}$$

The polynomial $\Phi(w, z)$ is said to be *reducible* or *irreducible* depending on whether it is possible or impossible to represent it as the product $\Phi_1(w, z) \cdot \Phi_2(w, z)$, where $\Phi_1(w, z)$ and $\Phi_2(w, z)$ are polynomials in the variables w and z which are not equal to a constant. In the case, where in fact $\Phi(w, z) = \Phi_1(w, z) \cdot \Phi_2(w, z)$ and l, l_1, l_2 are the degrees of the corresponding polynomials, $l = l_1 + l_2$.

Let us note that for an irreducible polynomial $\Phi(w, z)$ the greatest common factor of the coefficients $\phi_0(z), \phi_1(z), \dots, \phi_n(z)$ is a constant. In the contrary case the polynomial $\Phi(w, z)$ could be represented as $\Phi_1(z) \cdot \Phi_2(w, z)$, where $\Phi_1(z)$ is a polynomial of a certain degree $p > 0$ with respect to z and of zero degree with respect to w.

In what follows (the reason, will be indicated in § 5) we shall usually consider irreducible polynomials.

Example 1. The polynomials

$$F_1 = w^2 - z^2$$

$$F_2 = zw^3 + z^2$$

are reducible, as

$$w^2 - z^2 = (w - z)(w + z)$$

$$zw^3 + z^2 = z(w^3 + z).$$

Example 2. The polynomial

$$\Phi = w^2 + z + 1$$

is irreducible. In fact, the polynomial, by which it could be divided giving as quotient another polynomial, different from a constant, must have the form $aw + bz + c$ (where a and b are not simultaneously zero). However, it is easily seen that the remainder in the division of the polynomial $w^2 - z + 1$ by the polynomial $aw + bz + c$ fulfilling the conditions stated is always different from zero.

2. The resultant of two polynomials

Let us consider the system of two equations:

$$\begin{aligned} \Phi(w, z) &= \phi_0(z)w^n + \phi_1(z)w^{n-1} + \dots + \phi_n(z) = 0, \\ \Psi(w, z) &= \psi_0(z)w^p + \psi_1(z)w^{p-1} + \dots + \psi_p(z) = 0. \end{aligned} \tag{1.5}$$

We shall search for that value $z = \beta$, for which both the equations (1.5) can be satisfied for one and the same value $w = \alpha$. In other words we want

$$w = \alpha, \quad z = \beta$$

to be a solution of the system of equations (1.5).

For this purpose let us form the auxiliary system of equations:

$$\left.\begin{array}{rl}
w^{p-1}\Phi = & \phi_0 w^{n+p-1} + \phi_1 w^{n+p-2} + \dots + \phi_n w^{p-1} = 0,\\
w^{p-2}\Phi = & \phi_0 w^{n+p-2} + \dots + \phi_{n-1} w^{p-1} + \phi_n w^{p-2} = 0,\\
\dots & \dots\\
\Phi = & \phi_0 w^n + \dots\dots + \phi_n = 0,\\
w^{n-1}\Psi = & \psi_0 w^{n+p-1} + \psi_1 w^{n+p-2} + \dots + \psi_p w^{n-1} = 0,\\
w^{n-2}\Psi = & \psi_0 w^{n+p-2} + \dots + \psi_{p-1} w^{n-1} + \psi_p w^{n-2} = 0,\\
\dots & \dots\\
\Psi = & \psi_0 w^p + \dots\dots + \psi_p = 0.
\end{array}\right\} \quad (1.6)$$

If the values $w = \alpha$, $z = \beta$ satisfy the equations $\Phi = 0$ and $\Psi = 0$ then they, obviously, must satisfy all the equations of the system (1.6).

Also let us consider another auxiliary system of $n+p$ homogeneous linear equations with $n+p$ unknowns $x_0, \dots, x_{n+p-1}$.

$$\left.\begin{array}{r}
\phi_0(\beta)x_{n+p-1} + \phi_1(\beta)x_{n+p-2} + \dots + \phi_n(\beta)x_{p-1} = 0,\\
\phi_0(\beta)x_{n+p-2} + \dots + \phi_{n-1}(\beta)x_{p-1} + \phi_n(\beta)x_{p-2} = 0,\\
\dots\dots\dots\\
\phi_0(\beta)x_n + \dots\dots + \phi_n(\beta)x_0 = 0,\\
\psi_0(\beta)x_{n+p-1} + \psi_1(\beta)x_{n+p-2} + \dots + \psi_p(\beta)x_{p-1} = 0,\\
\psi_0(\beta)x_{n+p-2} + \dots + \psi_{p-1}(\beta)x_{p-1} + \psi_p(\beta)x_{p-2} = 0,\\
\dots\dots\dots\\
\psi_0(\beta)x_p + \dots\dots + \psi_p(\beta)x_0 = 0.
\end{array}\right\} \quad (1.7)$$

As the system of equations (1.6) is satisfied by the values $w = \alpha$, $z = \beta$, the system of homogeneous linear equations (1.7) has the set of solutions

$$x_{n+p-1} = \alpha^{n+p-1}, \quad x_{n+p-2} = \alpha^{n+p-2}, \dots, x_0 = 1. \qquad (1.8)$$

Therefore, the $n+p$ homogeneous linear equations (1.7) with $n+p$

unknowns become identities on the substitution into them of the values (1.8) (of which one is certainly different from zero, as $x_0 = 1$). This is possible only in the case where the determinant of the coefficients of this system is equal to zero. In other words, the value $z = \beta$ reduces to zero the determinant

$$R(\Phi, \Psi) = \left|\begin{array}{cccccccc} \phi_0(z) & \phi_1(z) & \ldots & \phi_n(z) & 0 & \ldots & 0 \\ 0 & \phi_0(z) & \ldots & \phi_{n-1}(z) & \phi_n(z) & \ldots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \ldots & \phi_0(z) & \ldots & \ldots & \phi_n(z) \\ \psi_0(z) & \psi_1(z) & \ldots & \psi_p(z) & 0 & \ldots & 0 \\ 0 & \psi_0(z) & \ldots & \psi_{p-1}(z) & \psi_p(z) & \ldots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \ldots & \psi_0(z) & \ldots & \ldots & \psi_p(z) \end{array}\right| \begin{array}{l} \left.\vphantom{\begin{array}{c}a\\a\\a\\a\end{array}}\right\} p \\ \left.\vphantom{\begin{array}{c}a\\a\\a\\a\end{array}}\right\} n \end{array} \qquad (1.9)$$

The expression $R(\Phi, \Psi)$ is a polynomial with respect to the variable z. We have shown, that *only for values of* $z = \beta$, *satisfying the equation*

$$R(\Phi, \Psi) = 0, \qquad (1.10)$$

can there exist values of w, *simultaneously satisfying the equations*

$$\Phi(w, \beta) = 0, \qquad \Psi(w, \beta) = 0. \qquad (1.11)$$

At the same time it does not in general follow from our discussion, that for values of $z = \beta$, which are roots of the equation (1.10), there actually exist values of w, simultaneously satisfying both of equations (1.11).†

In courses on higher algebra it is established that equations (1.5) always have at least one common root $w = \alpha$ at points z, where $R(\Phi, \Psi) = 0$, and the highest coefficients $\phi_0(z)$ and $\psi_0(z)$ are different from zero.‡ Equation (1.10) is called the *resultant* of the equations (1.5) on the elimination of the unknown w. We will collect our results in the following proposition.

† From our discussion it follows, that to a value $z = \beta$, satisfying the equation (1.10), there correspond values of the unknowns $x_0, \ldots, x_{n+p-1}$ (not all equal to zero), reducing the equations (1.7) to identities. However, these values of the unknowns $x_0, \ldots, x_{n+p-1}$ could not be consecutive powers of a certain number α.

‡ See for example, Kurosh, A. G., *Course of Higher Algebra* (Moscow, Gostekhizdat, 1950), page 227.

THEOREM 1. *The value* $z = \beta$ *is a root of the resultant* (1.9) *of the two polynomials* (1.5), *and only if the equations*

$$\Phi(w, \beta) = 0, \qquad \Psi(w, \beta) = 0$$

have a common root $w = \alpha$ *or if*

$$\phi_0(\beta) = \psi_0(\beta) = 0.$$

Remark. In courses on higher algebra it is proved, that if l is the degree of the polynomial $\Phi(w, z)$, and l_2 is the degree of the polynomial $\Psi(w, z)$, then the degree of the polynomial $R(\Phi, \Psi)$ will not be higher than the product $l_1 l_2$.

Example 1. Solve the system of equations

$$\Phi = w^3 z + z - 3 = 0,$$
$$\Psi = \pm\sqrt[3]{(2)}w^2 + z - 3 = 0.$$

Solution

Construct the resultant of the given polynomials. By formula (1.9)

$$R(\Phi, \Psi) = \begin{vmatrix} z & 0 & 0 & z-3 & 0 \\ 0 & z & 0 & 0 & z-3 \\ \sqrt[3]{2} & 0 & z-3 & 0 & 0 \\ 0 & \sqrt[3]{2} & 0 & z-3 & 0 \\ 0 & 0 & \sqrt[3]{2} & 0 & z-3 \end{vmatrix} =$$
$$= (z-1)(z-3)^2(z-1+\sqrt{3})(z-1-\sqrt{3}).$$

Finding z from the equation $R(\Phi, \Psi) = 0$ and searching the given system of equations for those values of z, we find, that for $z = 1$ the equations $\Phi = 0$ and $\Psi = 0$ have the common solution $w = \sqrt[3]{2}$, for $z = 3$—the common solution $w = 0$, for $z = 1-\sqrt{3}$—the common solution $w = [\sqrt[3]{(2)}/(1-\sqrt{3})]$, for $z = 1+\sqrt{3}$—the common solution $w = [\sqrt[3]{(2)}/(1+\sqrt{3})]$. Thus we obtain the solutions of the given system of equations.

Example 2. *Solve the system of equations*

$$\Phi = w^2 z - w = 0,$$
$$\Psi = 2w^2 z + z - 3 = 0.$$

Solution

Let us construct the resultant of the polynomials Φ and Ψ. By formula (1.9)

$$R(\Phi, \Psi) = \begin{vmatrix} z & -1 & 0 & 0 \\ 0 & z & -1 & 0 \\ 2z & 0 & z-3 & 0 \\ 0 & 2z & 0 & z-3 \end{vmatrix} =$$

$$= z(z-1)(z-2)(z-3).$$

Finding z from the equation $R(\Phi, \Psi) = 0$ and then considering the given system of equations for those values of z, we find, that for $z = 3$ our equations have the common solution $w = 0$, for $z = 2$ —the common solution $w = \frac{1}{2}$, for $z = 1$—the common solution $w = 1$. For $z = 0$ the coefficients of the highest powers in the equations $\Phi = 0$ and $\Psi = 0$ become zero. For this value of z the given equations do not have a common root w.

3. The discriminant of a polynomial. Definition of an algebraic function

Let us consider the equation

$$\Phi(w, z) = \phi_0(z)w^n + \phi_1(z)w^{n-1} + \ldots + \phi_n(z) = 0. \quad (1.12)$$

We assume that $\phi_0(z) \neq 0$ and that the integer m is independent of the degree of the polynomials $\phi_0(z), \ldots, \phi_n(z)$.

Fixing the variable z, we transform (1.12) into an equation with constant coefficients, generally speaking, of the n-th degree with respect to w. If β_s are the roots of $\phi_0(z)$, then for $z = \beta_s$ this degree will be less than n.† We will suppose initially, that the fixed value of z which we have taken is not equal to one of the given roots β_s, and hence for it the degree of equation (1.12) is exactly equal to n. By

† Not excluding even the possibility, that at certain points $z = \beta_s$ the degree of equation (1.12) (with respect to w) may turn out to be equal to zero. This may happen, if at some point $z = \beta_s$

$$\phi_0(\beta_s) = \ldots = \phi_{n-1}(\beta_s) = 0;$$

then, if $\phi_n(\beta_s) \neq 0$, the equation $\Phi(w, \beta_s) = 0$ does not have finite roots; if $\phi_n(\beta_s) = 0$, the equation $\Phi(w, \beta_s)$ becomes an identity, satisfied for any value of w. In the last case all the $\phi_j(z)$ are divisible by $z - \beta_s$ (this cannot happen if the polynomial $\phi(w, z)$ is irreducible).

the fundamental theorem of algebra there must exist roots of this equation.† They may be all simple, that is, have multiplicity equal to one; then their total number is equal to n. If however they include multiple ones, the total number of the distinct roots w_s at this point will be less than n.

As is well known, if ω is a root of multiplicity q ($q \geqslant 1$) of some analytic function (in particular, a polynomial $f(w)$, then

$$f(\omega) = f'(\omega) = \ldots = f^{(q-1)}(\omega) = 0, \quad f^{(q)}(\omega) \neq 0. \quad (1.13)$$

Hence, turning to the case of equation (1.12), we can assert, that all its multiple roots (i.e. roots of multiplicity $q \geqslant 2$), must, apart from equation (1.12) itself, also satisfy at least the equation

$$\Phi'_w(w, z) = n\phi_0(z)w^{n-1} + (n-1)\phi_1(z)w^{n-2} + \ldots + \phi_{n-1}(z) = 0. \quad (1.14)$$

Thus, if for a certain $z = \beta$ equation (1.12) has the multiple root $w = \omega$, the values

$$w = \omega, \qquad z = \beta$$

form a solution of the set of equations (1.12) and (1.14). Consequently, the value $z = \beta$ must by theorem 1 satisfy the equation

$$D(z) = R(\Phi, \Phi'_w) = 0. \quad (1.15)$$

By theorem 1 the values of z, which reduce to zero the highest coefficients of the polynomials $\Phi(w, z)$ and $\Phi'(w, z)$, that is all the roots of the polynomial $\phi_0(z)$ also satisfy this equation.

The polynomial $D(z)$ is called the *discriminant* of the polynomial‡ $\phi(w, z)$. We shall denote its roots by the symbols β_s (the roots β_s of the polynomial $\phi_0(z)$, which are all roots of the polynomial $D(z)$, correspond to the first of our numbers s; the remaining roots of the polynomial $D(z)$ will now correspond to the following numbers s). From the preceding analysis it follows, that for the values $z = \beta_s$ and only for those (if we are speaking about finite values of z) the equation $\Phi(w, z) = 0$ can have less than n distinct roots.

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† See F.C.V., Chap. VII Art. 78, Ex. 3.

‡ We omit from the discussion the possibility that the discriminant $D(z)$ may become identically zero. This case can arise only if the polynomial $\Phi(w, z)$ has a factor of the form $[\phi(w, z)]^k$ where $\phi(w, z)$ is a certain polynomial and k is an integer, greater than unity. In particular, the discriminant of an irreducible polynomial cannot be identically equal to zero.

Let us extend the domain in which equation (1.12) is considered, by adding to it the point $z = \infty$. For this purpose we will put in (1.12) $z = z_1^{-1}$. Then

$$\Phi(w, z) = \Phi(w, z_1^{-1}) = z_1^{-m}\Psi(w, z_1). \tag{1.16}$$

Hence, in any neighbourhood of the value $z_1 = 0$ (the point $z_1 = 0$ itself is excluded from it) the equation $\Phi(w, z) = 0$ is equivalent to the equation $\Psi(w, z_1) = 0$. Let us agree to consider the equation

$$\Psi(w, 0) = 0 \tag{1.17}$$

as the result of continuing equation (1.12) at the point $z = \infty$. The roots $w_1, w_2, \ldots$ of equation (1.17) are considered as the roots of equation (1.12) corresponding to the value $z = \infty$.

The number of the distinct roots of equation (1.17) is determined just the same as before. In connexion with this, let us note that the discriminant $D_1(z_1)$ of the polynomial $\Psi(w, z)$, is obviously obtained from the discriminant $D(z)$ of the polynomial $\Phi(w, z)$ also with the help of the substitution $z = z_1^{-1}$ and subsequent multiplication by some power of z_1 (which happens to be in the denominator of the expression for $D(z)$).

The points of the complete z-plane, at which the number of distinct finite roots of equation (1.12) is less than n (where for the point $z = \infty$ equation (1.12) is replaced by equation (1.17)), are the "critical" points of the equation (1.2). As we have seen, every critical point is a root of $D(z) = 0$ ($z = \infty$, of course, is only a critical point if $D_1(0) = 0$).

We will collect our results in the following proposition.

THEOREM 2. *Given the equation*

$$\Phi(w, z) = \phi_0(z)w^n + \ldots + \phi_n(z) = 0,$$

where $\phi_0(z), \ldots \phi_n(z)$ are polynomials with respect to z and $\phi_0(z) \neq 0$. Let $\mathscr{E}$ be the domain, obtained by removing from the complete plane the critical points of the given equation. Then at every point $z \in \mathscr{E}$† *this equation has n distinct finite roots $w_1, \ldots, w_n$.*

Thus, in the domain $\mathscr{E}$ there is defined an n-valued function $w = w(z)$. We shall call it the algebraic function, determined by the

† The symbol $\in$ both here and in what follows denotes the symbol of inclusion. The notation $z \in \mathscr{E}$ indicates that the point z belongs to the domain $\mathscr{E}$.

equation $\Phi(w, z) = 0$. The points of the complete z-plane, not belonging to the domain $\mathscr{E}$, are called singular points of this algebraic function.

Example. *Find the domain $\mathscr{E}$ for the algebraic function, determined by the equation*

$$\Phi(w, z) = w^3z^2 - wz + 1 = 0.$$

Solution

Let us take the equation

$$\Phi'_w(w, z) = 3w^2z^2 - z = 0$$

and construct by formula (1.9) the discriminant $D(z) = R(\Phi, \Phi_w')$. We find, that

$$D(z) = -z^6(4z - 27).$$

Thus, to the domain $\mathscr{E}$ belong all the finite points of the z-plane except for the points

$$\beta_1 = 0, \quad \beta_2 = \frac{27}{4}.$$

At the point $z = 0$ in the polynomial $\Phi(w, z)$ the coefficient of w^3 (equal to z^2) reduces to zero. At the point $z = 27/4$ the equation $\Phi(w, z) = 0$ has only two distinct roots: $w = -4/9$ and $w = 2/9$.

In order to decide whether the point $z = \infty$ belongs to the domain $\mathscr{E}$ we have to examine at the point $z_1 = 0$ the equation obtained from the given one by the substitution $z = z_1^{-1}$:

$$\Psi(w, z_1) = w^3 - wz_1 + z_1^3 = 0.$$

It is obvious, that at the point $z_1 = 0$ this equation has only the one root $w = 0$. Thus, the point $z = \beta_3 = \infty$ is the third point, which must be excluded from the complete z-plane in order to obtain the domain $\mathscr{E}$ of the given equation. We should have arrived at the same conclusion by considering the discriminant of the last equation at the point $z_1 = 0$.

4. The existence of regular branches of algebraic functions

In the present article we shall prove the following important proposition.

THEOREM 3. *Let $\Phi(w, z)$ be a polynomial in two variables w and z. If for the values $w = w_0$, $z = z_0$*

$$\Phi(w_0, z_0) = 0, \quad \Phi'_w(w_0, z_0) \neq 0 \quad (\text{for } z_0 \neq \infty), \tag{1.18}$$

or

$$\Psi(w_0, 0) = 0, \quad \Psi'_w(w_0, 0) \neq 0 \quad (\text{for } z_0 = \infty) \tag{1.19}$$

(where in both cases $w_0 \neq \infty$), then in the neighbourhood of the point $z = z_0$ there exists one and only one regular function $w = w(z)$, such that

$$w(z_0) = w_0 \quad \textit{for} \quad z_0 \neq \infty, \tag{1.20}$$

or

$$\lim_{z_1 \to 0} w(z_1^{-1}) = w_0 \quad \textit{for} \quad z_0 = \infty. \tag{1.21}$$

and

$$\Phi(w(z), z) \equiv 0. \tag{1.22}$$

Remark. The condition $\Phi'_n(w_0, z_0) \neq 0$ (or $\Psi'_n(w_0, 0) \neq 0$, if $z_0 = \infty$) is satisfied, if the point $z_0 \in \mathscr{E}$. The theorem formulated by us is always applicable in this case.

Proof. Let us consider first the case when $z_0 \neq \infty$. In order to simplify the further calculations let us take the quantities

$$w - w_0, \qquad z - z_0 \tag{1.23}$$

as new variables. Thus, the proof reduces to the case, where $w_0 = z_0 = 0$. We will retain the previous notation for the new variables. Thus, we shall suppose, that $\Phi(0, 0) = 0$, $\Phi'_n(0, 0) \neq 0$. We have to prove that there exists one and only one function $w(z)$ regular to the point $z = 0$, satisfying the conditions

$$w(0) = 0, \quad \Phi(w(z), z) \equiv 0. \tag{1.24}$$

Bearing in mind that $\Phi(0, 0) = 0$, $\Phi'_n(0, 0) \neq 0$, we shall represent the equation $\Phi(w, z) = 0$ in the following form:

$$w = F(w, z) = a_{01}z + a_{20}w^2 + a_{11}wz + a_{02}z^2 + \dots \tag{1.25}$$

Actually, thanks to the fact that $\Phi(0, 0) = 0$, the absolute term in the polynomial $\Phi(w, z)$ is equal to zero; as $\Phi'_n(0, 0) \neq 0$, the coefficient of w in this polynomial is different from zero; hence the equation $\Phi(w, z) = 0$ can be represented in the form of equation (1.25).

Let us assume now, that the required function $w = w(z)$ has been found by us in the form of the series

$$w = c_1 z + c_2 z^2 + \dots \tag{1.26}$$

(we take into account the first condition of (1.24) and take the absolute term in the series (1.26) to be equal to zero). Then by the second condition of (1.24) we obtain as a result of the substitution of the series (1.26) in the equation (1.25) the identity

$$\begin{aligned} c_1 z + c_2 z^2 + \dots &= a_{01} z + a_{20}(c_1 z + c_2 z^2 + \dots)^2 + \\ &+ a_{11} z (c_1 z + c_2 z^2 + \dots) + a_{02} z^2 + a_{30}(c_1 z + c_2 z^2 + \dots)^3 + \dots \end{aligned} \tag{1.27}$$

Equation (1.27) holds for all the values of z, lying in a certain neighbourhood of the origin of co-ordinates. Hence it follows, that the coefficients of the same powers of z_1 on both sides of this identity are equal to one another. Thus, we obtain the relations†

$$\begin{gathered} c_1 = a_{01}, \qquad c_2 = a_{20} c_1^2 + a_{11} c_1 + a_{02}, \\ c_3 = 2 a_{20} c_1 c_2 + a_{11} c_2 + a_{30} c_1^3 + a_{21} c_1^2 + a_{03}, \\ \dots\dots\dots\dots\dots\dots \end{gathered} \tag{1.28}$$

Thus, we successively find that

$$\begin{aligned} c_1 &= a_{01}, \quad c_2 = a_{20} a_{01}^2 + a_{11} a_{01} + a_{02}, \\ c_3 &= 2 a_{20} a_{01} (a_{20} a_{01}^2 + a_{11} a_{01} + a_{02}) + \\ &\quad + a_{11} (a_{20} a_{01}^2 + a_{11} a_{01} + a_{02})^2 + a_{30} a_{01}^2 + a_{03}, \\ &\dots\dots\dots\dots\dots\dots \end{aligned} \tag{1.29}$$

Thus, if the series (1.26) satisfies the requirements of theorem 3, its coefficients have values determined by the equations (1.29). These equations uniquely determine the coefficients c_k; hence, there can exist only one function, satisfying the conditions of theorem 3.

However, the discussion which we have carried out does not prove the existence of even one such function: this fact was its initial assumption. In order to examine this assertion let us consider the series:

$$w = a_{01} z + (a_{20} a_{01}^2 + a_{11} a_{01} + a_{02}) z^2 + \dots \tag{1.30}$$

† The power series (1.26) converges absolutely in its circle of convergence. On the right-hand side of equation (1.25) there stands a polynomial. Hence it follows that the operations, performed in order to obtain the relations (1.28), are justified.

Its coefficients are quantities, determined by the formulas (1.29). If the series (1.30) is convergent in some circle $|z| < r$, then the function represented by it will be the one sought. Actually, as the sum of a power series it will be regular. There is no doubt about the satisfaction of the first condition of (1.24). The satisfaction of the second condition of (1.24) follows from the fact that for the series (1.30) the relation (1.28) becomes an identity.

We will derive the proof of the convergence of the series (1.30) by the method of comparison functions (majorant) of Cauchy, which is widely applied in the whole of mathematical analysis. The polynomial (1.25) is the sum of a finite number of monomials of the form $a_{kl}w^k z^l$. Let us denote by M the greatest value of the moduli of all these monomials for $|w| \leqslant R, |z| \leqslant R_1$, where R and R_1 are certain arbitrary positive numbers. Then for all the indices l and k present in the polynomial (1.25),

$$|a_{kl}| \leqslant \frac{M}{R^k R_1{}^l}. \tag{1.31}$$

We consider together with equation (1.25) the auxiliary equation

$$W = \frac{M}{\left(1-\dfrac{W}{R}\right)\left(1-\dfrac{z}{R_1}\right)} - M - \frac{M}{R_1}z \equiv$$

$$\equiv \alpha_{01}z + \alpha_{20}W^2 + \alpha_{11}Wz + \alpha_{02}z^2 + \dots \tag{1.32}$$

The last expression is an infinite series (easily obtained by the multiplication of the geometrical progressions with ratios w/R and z/R_1). In it

$$\alpha_{kl} = \frac{M}{R^k R_1{}^l} \geqslant |a_{kl}|. \tag{1.33}$$

The relation (1.32) is a quadratic equation with respect to W. As is shown by calculation, its unique solution, satisfying the condition $W(0) = 0$, is the function

$$W = W(z) = \frac{R^2}{2(R+M)}\left[1 - \sqrt{\frac{1-(z/R_2)}{1-(z/R_1)}}\right], \tag{1.34}$$

where

$$R_2 = R_1\left(\frac{R}{R+2M}\right)^2. \tag{1.35}$$

In formula (1.34) the branch of the quadratic root is taken, the argument of which is greater than (or equal to) 0 and less than π.†

Let us expand the function (1.34) into a series of powers of z. This representation can be obtained by commencing with the expansions of the functions $[1-(z/R_1)]^{-\frac{1}{2}}$ and $[1-(z/R_2)]^{\frac{1}{2}}$ in binomial series; finally it is necessary to carry out the operation indicated in formula (1.34). As a result we arrive at the series

$$W(z) = \gamma_1 z + \gamma_2 z^2 + \gamma_3 z^3 + \dots \tag{1.36}$$

This, like the original binomial series‡ will converge for $|z| < R_2$.

The series (1.36) defines the solution of equation (1.32). Hence, repeating the reasoning which led us to equation (1.29) we obtain

$$\gamma_1 = \alpha_{01}, \qquad \gamma_2 = \alpha_{20}\alpha_{01}^2 + \alpha_{11}\alpha_{01} + \alpha_{02}, \tag{1.37}$$
$$\cdots\cdots\cdots\cdots$$

Hence, using equation (1.33), we find that the coefficients of the series (1.30) and (1.36) are connected by the relations

$$|a_{01}| \leqslant \alpha_{01} = \gamma_1,$$
$$|a_{20}a_{01}^2 + a_{11}a_{01} + a_{02}| \leqslant |a_{20}|\,|a_{01}|^2 + |a_{11}|\,|a_{01}| + |a_{02}| \leqslant$$
$$\leqslant \alpha_{20}\alpha_{01}^2 + \alpha_{11}\alpha_{01} + \alpha_{02} = \gamma_2, \tag{1.38}$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$

Hence, the moduli of the coefficients of the power series (1.30) turn out to be less than the coefficients of the power series (1.36), convergent in the circle $|z| < R_2$. Hence by the principle of the comparison of series it follows that the series (1.30) also converges in the circle $|z| < R_2$.

Therefore, our theorem is proved for the case where $z_0 \neq \infty$. Our reasoning is easily extended to the point $z_0 = \infty$ also. Here it is necessary, just as at the end of the preceding article, to pass with the help of the substitution $z = z^{-1}$, from the given equation $\Phi(w, z) = 0$ to the equation $\Psi(w, z) = 0$ and then make use of the already proved case of theorem 3 for $z_1 = 0$. As a result in place of the series (1.30) we obtain a series, consisting of negative powers of the variable z.

Remark. Let us pause again on the method of proof of theorem 3. It is based on the use of two methods:

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† See F.C.V., Chap. III, Art. 24.

‡ See F.C.V., Chap. VI, Art. 61, formula (21).

1. At the beginning of the proof the "method of undetermined coefficients" is used. We substitute into the equation considered the series (1.26) and determine the coefficients c_k of this series. As a result we arrive at the series (1.30).

2. Then there is introduced the comparison function (1.32). The coefficients of the series (1.32) are positive and greater than the moduli of the corresponding coefficients of the polynomial (1.25). The coefficients of the power series (1.30), determining the solution of the given equation (1.25), are formed from the coefficients of the latter by the formulas (1.29) by means of addition and multiplication. Hence on replacing the given equation by equation (1.32) we obtain as solution the series (1.36) with positive coefficients, with greater moduli than the corresponding coefficients of the series (1.30). Then from the convergence of the series (1.36), the convergence of the series (1.30) follows.

Let us note also, that the preceding reasoning can without any kind of material alteration be applied to an arbitrary analytic function $\Phi(w, z)$ of two complex variables w and z, that is to the case, where instead of the polynomial (1.25) there is considered an infinite power series. Above we have limited ourselves only to the particular case necessary for what follows.

From theorem 3 it follows, that in a certain neighbourhood $D_0 \subset \mathscr{E}$† of the arbitrary point $z_0 \in \mathscr{E}$ there exist n and only n regular functions $w_k(z)$, satisfying the equation $\Phi(w, z) = 0$. Then

$$w_k(z_0) = w_k \qquad (k = 1, 2, \ldots, n), \tag{1.39}$$

where the w_k are the roots of the equation $\Phi(w, z_0) = 0$. These functions $w_k(z)$ are called the regular branches of the algebraic function $w(z)$, determined by the equation $\Phi(w, z) = 0$.

5. The analytic continuation of the regular branches of an algebraic function

The regular branches $w_k(z)$ of the many-valued function $w(z)$ are determined independently of one another in the neighbourhood of all the points of the domain $\mathscr{E}$. In the present article there will be established the connexion between these regular branches.

Let the respective neighbourhoods of the points $z_1 \in \mathscr{E}$ and $z_2 \in \mathscr{E}$—the circles $D_1[|z-z_1| < \rho_1]$ and $D|_2[z-z_2| < \rho_2]$—overlap

† The notation $D_0 \subset \mathscr{E}$ denotes, that D_0 is a subset of the set $\mathscr{E}$. The symbol $\subset$ will be used with the same meaning in what follows.

one another in a certain domain D (Fig. 1; it is here assumed that $D_1 \subset \mathscr{E}, D_2 \subset \mathscr{E}$).

Let $w_k^{(1)}(z)$ and $w_k^{(2)}(z)$, $k = 1, 2, \ldots n$, be regular branches of the algebraic function $w(z)$ in the neighbourhoods D_1 and D_2. In the neighbourhood of every point $z_3 \in D$ there exist n and only n regular branches of the function $w(z)$, satisfying the equation $\Phi(w, z) = 0$. Hence the functions $w_k^{(1)}(z)$ and $w_k^{(2)}(z)$ must be identical in pairs in the domain D, that is we have there

$$w_k^{(1)}(z) = w_k^{(2)}(z) \qquad (k = 1, 2, \ldots, n). \tag{1.40}$$

Let us note, that in this it may be necessary to change the numbering

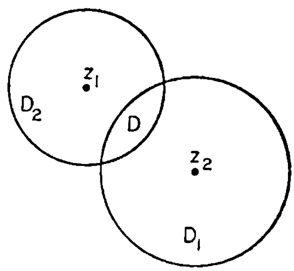

FIG. 1

of one of these systems, for example of the functions $w_k^{(2)}(z)$. *Therefore, the functions $w_k^{(2)}(z)$ turn out to be direct analytic continuations of the corresponding functions $w_k^{(1)}(z)$ from the circle D_1 into the circle D_2.* We arrive at a similar result in the case where one of the points z_1 or z_2 is the point at infinity.

Also let z' and z'' be two arbitrary points of the domain $\mathscr{E}$, D' and D'' be circles with centres at these points. We will connect† these circles by a chain of circles D_α with centres at certain points $z_\alpha \in \mathscr{E}$, where $\alpha = 1, \ldots \nu$. Here $z_1 = z'$, $z_\nu = z''$; the circle D_1 is identical with the circle D', the circle D_ν with the circle D''; the circles D_α and $D_{\alpha+1}$ for all $\alpha = 1, \ldots, \nu-1$ overlap one another (Fig. 2). Finally, the radii of the circles D_α (in particular, of the circles D' and D'') are assumed to be so small, that these circles are contained within the domain $\mathscr{E}$, and in each of them there is determined (by the method indicated in theorem 3) n regular branches

† The further exposition of the present article is of a descriptive character; a strict exposition of the questions considered here would take us outside the range of this book.

$w_k^{(\alpha)}(z)(k = 1, \ldots n)$ of the algebraic function $w(z)$. Let us note also, that if any of the points z_α is identical with the point at infinity, for the domain D_α it is necessary to take the circle $|z| > R$, where R is a sufficiently large number.

Let us introduce the notation:

$$w_k^{(1)}(z) = w_k(z), \quad w_k^{(\nu)}(z) = W_k(z) \quad (k = 1, \ldots, n). \qquad (1.41)$$

Then, repeating the preceding reasoning the necessary number of times, we establish, that the functions $w_k^{(\alpha)}(z)$ and, in particular, the functions $W_k(z)$ are obtained from the functions $w_k(z)$ as a result of analytic continuation along the chain of circles D_α. Insofar as the

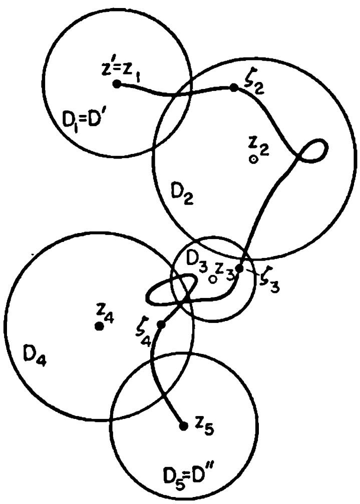

FIG. 2

functions $w_k^{(\alpha)}(z)(k = 1, \ldots, n)$ were originally defined in the circles D_α independently of one another, the subscripts of the functions, which turn out to be analytic continuations of one another in the various circles D_α, will not coincide in general with one another. However, such a coincidence is easily attained by introducing the necessary changes in the numbering of the functions $w_k^{(\alpha)}(z)$.

Thus, the process of analytic continuation is defined in the domain $\mathscr{E}_1$, by constructing a chain of circles D_α, and defining n regular functions $w_1(z), \ldots w_n(z)$, in such a way, that in each circle D_α, $w_\kappa(z) = w_k^{(\alpha)}(z)$.

Let us take in each circle D_α the point ζ_α, and suppose that $\zeta_1 = z'$, $\zeta_\nu = z''$. Then let us construct the piecewise smooth curve L not possessing multiple points, which connects the points z' and z'' and passes through all the points ζ_α. It is assumed, that the segment of the curve L between the points ζ_α and $\zeta_{\alpha+1}$ lies entirely within the circles D_α and $D_{\alpha+1}$.

Let ζ be a point on the arc $\zeta_\alpha\zeta_{\alpha+1}$ of the curve L, lying within the intersection of the circles D_α and $D_{\alpha+1}$. The values of the functions $w_k^{(\alpha)}(z)$ and $w_k^{(\alpha+1)}(z)$ $(k = 1, \ldots, n)$ coincide with one another in pairs, whereby

$$w_k^{(\alpha)}(\zeta) = w_k^{(\alpha+1)}(\zeta) = \omega_k \qquad (k = 1, \ldots, n). \tag{1.42}$$

Here the ω_k are the roots of the equation $\Phi(w, \zeta) = 0$. As $\zeta \in \mathscr{E}$ all the ω_k are distinct. Hence only one of the functions

$$w_1^{(\alpha+1)}(z), \ldots, w_n^{(\alpha+1)}(z),$$

namely the function $\omega_k^{(\alpha+1)}(z)$, turns out to be a continuous continuation of the function $w_k^{(\alpha)}(z)$ at the point ζ. This circumstance permits us to replace the process of analytic continuation of the functions $w_\kappa(z)$ along the chain of circles D_α from the point z' to the point z'' by the process of the continuous continuation of these functions along the curve L, connecting the point z' with the point z''. *In continuous continuation we have to take care only that the values of the functions* $w_k^{(\alpha)}(z)(\alpha = 1, \ldots, \nu)$ *continuously adjoin one another* (in the passage from one circle D_α to another).

The value of the function $w_\kappa(z)$ at the point z'' obtained as a result of such a continuation defines uniquely, by theorem 3, this function in the neighbourhood of the point z''. The same thing holds for every other point of the curve L.

It is easy to establish, from the uniqueness theorem of the theory of regular functions, that the continuous continuation of our functions along the curve L is equivalent to their analytic continuation along any other chain of circles D_α^*, with respect to which the curve L can be defined by the method given above. We shall not stop to prove this assertion.

Let us now consider the piecewise smooth curves L_1 and L_2, consisting of points of the domain $\mathscr{E}$ and connecting the point z' to the point z''. As a result of the continuation of the functions $w_\kappa(z)$ from the neighbourhood of the point z' to the neighbourhood of the point z'' along the curves L'_1 and L_2 we shall obtain there one

and the same set of functions $W_k(z)$ (in the circle D'' there do not exist other regular functions, satisfying the equation $\Phi(w, z) = 0$). However the rule of their correspondence with the original functions $w_k^{(1)}(z)$ in the continuation along the paths L_1 and L_2 may turn out to be different. For example, continuing the function $w_1^{(1)}(z)$ along the path L_1, we may obtain in the circle D'' the function $W_s(z)$, and continuing it along the path L_2, the function $W_t(z)$.

Let us suppose, that, in fact, continuing the function $w_1^{(1)}(z)$ along the line L_1, we obtain in the circle D'' the function $W_1(z)$. Then we commence to continue the function $W_s(z)$ along the line L_2 from the point z'' to the point z'. In the continuation along this line the function $w_1^{(1)}(z)$ in the circle D' corresponds to the original function $W_t(z)$ in the circle D''; therefore, the result of the continuation of the function $W_s(z)$ along L_2 into the circle D' turns out to be some other function $w_k^{(1)}(z)$ (where $k \neq 1$). Therefore, in the case considered the continuation of the function $w_1^{(1)}(z)$ along the closed contour $\Lambda = L_1 + L_2$ leads us to a different function $w_k^{(1)}(z)$ $(k \neq 1)$. *The simultaneous continuation of all the functions $w_k^{(1)}(z)$ along the contour Λ leads to a certain rearrangement of the functions of this system.*

The question arises: *when, in continuing the function $w_k(z)$ along a certain closed contour Λ, shall we obtain the original function, or (what is equivalent) when do the continuations of the function $w_k^{(1)}(z)$ along two lines, joining the points z' and z'', give one and the same result?*

The solution of this question can be advanced to a significant extent by means of the deformation of the given contour. Let us replace the contour Λ by the contour Λ_1. Let us assume, that the contours Λ and Λ_1 satisfy the conditions, laid down for the construction of the path L with reference to one and the same chain of circles D_α. Then the result of continuing the functions $w_k^{(1)}(z)$ remains as previously, with suitable changes. Then let us take instead of the contour Λ_1, the contour Λ_2. We shall assume once more, that the contours Λ_1 and Λ_2 possess the above mentioned properties with respect to some chain of circles D_α. Then the result of continuing the functions $w_k^{(1)}(z)$ along the path Λ_2 is identical with the result of continuation along the path Λ_1, and consequently, also Λ.

We shall operate in exactly the same way later on. If as a result we can move the contour Λ to the contour Λ', lying as a whole in a circle D' (the continuation along this contour does not change the values of the function $w_k^{(1)}(z)$ after returning to the point z'), then

also the continuation of our functions along the contour Λ, obviously, returns each of them to its original value at the point z'.†

If it turns out to be impossible to move the contour Λ to the similar contour Λ', then the continuation of the functions $w_k(z)$ along the contour Λ, generally speaking, leads to their permutation.

More detailed investigation shows that only the points composing the discriminantal set of the equations $\Phi(w, z) = 0$, that is the points not belonging to the domain $\mathscr{E}$, can act as an obstacle to the passage of the contour Λ into the contour Λ'. In particular, the continuation of the function $w_k(z)$ along a contour, lying as a whole within a certain simply connected domain, contained in $\mathscr{E}$, always returns these functions to their original values.‡

Hence it is natural to expect, that the character of the permutation of the functions $w_k^{(1)}(z)$, which take place as a result of their continuation along the contour Λ, is determined by the distribution of the points of the discriminantal set of the equation on the sphere of the complex variable z.

Now let us consider the set of all such contours Λ, passing through a certain fixed point $z_0 \in \mathscr{E}$ (here the Λ, as also previously, are piecewise smooth, closed curves, lying as a whole inside the domain $\mathscr{E}$). We shall continue along these paths one of the regular branches $w_1^{(0)}(z)$ of our algebraic function, originally prescribed in a certain neighbourhood of the point z_0. What roots of the equation $\Phi(w, z) = 0$ can be obtained at the point z_0 as a result of such continuation? The answer to this question is given by the following proposition.

THEOREM 4. *Let $\Phi(w, z)$ be an irreducible polynomial in the variables w and z, $\mathscr{E}$ a domain, obtained by removing from the complete z-plane the discriminantal set of the equation $\Phi(w, z) = 0$, z_0 a certain point of the domain $\mathscr{E}$. Let the function $w_1^{(0)}(z)$ be regular at the point z_0 and satisfy in a certain neighbourhood of it the equation $\Phi(w, z) = 0$. Then by means of the continuation of the function $w_1^{(0)}(z)$ along the*

† The contour Λ' can, in particular, lie either within the contour Λ, or outside it. The reader is recommended to imagine to himself the contours Λ and Λ' on the sphere of the complex variable z.

‡ This fact constitutes a particular case of the theorem of monodromy which is extremely general and plays an important part in the theory of analytic functions.

contours Λ, *passing through the point* z_0 *and lying in the domain* $\mathscr{E}$, *there can be obtained all the remaining functions,* $w_k^{(0)}(z)$, *regular at the point* z_0 *and satisfying in its neighbourhood the condition*

$$\Phi(w_k^{(0)}(z), z) = 0. \tag{1.43}$$

Let us note, that theorem 4 refers only to irreducible polynomials, that is to polynomials, which cannot be represented in the form of the product of two other polynomials. Such a limitation is quite natural, as a reducible polynomial is the product of two or more polynomials which are quite independent of one another. The algebraic functions, which arise as a result of equating them to zero, do not generally speaking, have anything in common; there is no reason to expect that the continuation of the regular branches of one of these algebraic functions will lead to branches of the other, and conversely.

We shall not give the proof of theorem 4. In virtue of this proposition all the regular branches of such an algebraic function are analytic continuations of one another. No other kind of regular function can arise in their analytic continuation. Hence as a whole they determine in the domain $\mathscr{E}$ a complete (n-valued) analytic function, obviously identical with the algebraic function $w(z)$ introduced by us at the end of section 3. Thus, we have arrived at the result, that *the algebraic function* $w(z)$, *subject to the condition, that the polynomial* $\Phi(w, z)$ *is irreducible, is a complete analytic function in the domain* $\mathscr{E}$.

Example 1. Let us consider the algebraic function defined by the equation

$$\Phi = w^2 - (z-b_1)(z-b_2).$$

Here b_1 and b_2 are certain complex constants. As a result of calculation we find, that in this case

$$D(z) = R(\Phi, \Phi'_w) = 4(z-b_1)(z-b_2).$$

Thus, the points $z_1 = b_1$ and $z_2 = b_2$ belong to the discriminantal set of the given equation. Making the substitution $z = z_1^{-1}$, we establish that the point $z_2 = \infty$ must also belong to the discriminantal set. Therefore, the domain $\mathscr{E}$ for the given equation is that complete plane, from which the points b_1, b_2, ∞ have been removed.

At a certain point $z \in \mathscr{E}$ the roots of the given equation are determined by the formulas†

$$w_1(z) = \sqrt{(|z-b_1|\,|z-b_2|)}\; e^{i\frac{\arg(z-b_1)+\arg(z-b_2)}{2}}$$
$$w_2(z) = -w_1(z).$$

Each of the functions $\arg(z-b_s)(s = 1, 2)$ is continuous everywhere in the z-plane with the exception of the points of the ray

$$(R_s): \qquad \operatorname{Im}(z-b_s) = 0, \qquad -\infty < \operatorname{Re}(z-b_s) < 0.$$

In passing from the upper side of this ray to the lower this function

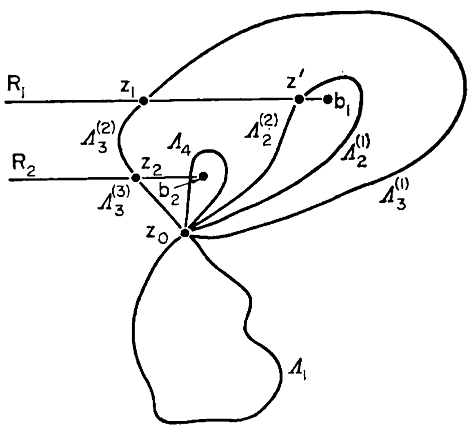

FIG. 3

receives an increment, equal to -2π. Hence at the points of the rays R_1 and R_2 the functions $w_k(z)$ also suffer a discontinuity—they change their sign.

Let us consider a certain point $z_0 \in \mathscr{E}$ and the contours Λ_1, Λ_2, Λ_3 passing through it. Let $\Lambda_2{}^{(1)}$, $\Lambda_2{}^{(2)}$ be segments of the contour Λ_2, let $\Lambda_3{}^{(1)}$, $\Lambda_3{}^{(2)}$, $\Lambda_3{}^{(3)}$ be segments of the contour Λ_3, into which Λ_2 and Λ_3 are subdivided by the point z_0 and by the points of intersection with the rays R_1 and R_2 (Fig. 3).

The contour Λ_1 is not intersected by the rays R_1 and R_2. Along it the functions $w_1(z)$ and $w_2(z)$ are continous and after traversing it they return to their original values.

† Here the symbol arg denotes the principal branch of the argument (F.C.V., Introduction, Art. 2. The symbol $\sqrt{\ }$ has its arithmetical meaning.

Now let us turn to the contour Λ_2. In the displacement of the point z along the line $\Lambda_2^{(1)}$ from the point z_0 to the point z' the function $w_1(z)$ constitutes its own continuous continuation. At the point z' the function $w_1(z)$ suffers a discontinuity. At the point z' the values of the function $w_2(z)$ on the line $\Lambda_2^{(2)}$ continuously follow on to the values of the function $w_1(z)$ on the line $\Lambda_2^{(1)}$. Also, on the line $\Lambda_2^{(2)}$ the function $w_2(z)$ is continuous, and hence as a result of the continuous continuation of the root of $w_1(z_0)$ along the contour Λ_2 we return to the point z_0 with the value of the continued root, equal to $w_2(z_0)$. On going once more round the contour Λ_2, we obtain instead of $w_2(z_0)$ the original root $w_1(z_0)$.

Finally, let us take the contour Λ_3. On displacing the point z along its segment $\Lambda_3^{(1)}$ from the point z_0 to the point z_1 the continuous continuation of the root $w_1(z_0)$ gives the same function $w_1(z)$. At the point z_1 the function $w_1(z)$ suffers a discontinuity. To the values of the function $w_1(z)$ on the line $\Lambda_3^{(1)}$ at the point z_1 there continuously follows on the values of the function $w_2(z)$ on the line $\Lambda_3^{(2)}$. On this segment the function $w_2(z)$ is a continuous continuation of the root $w_1(z_0)$. At the point z_2 the function $w_2(z)$ suffers a discontinuity. To its values on the line Λ_3 there continuously follow on at the point z_2 the values of the function $w_1(z)$ on the line $\Lambda_3^{(3)}$. Also, on the line $\Lambda_3^{(3)}$ the function $w_1(z)$ is continuous, and thus, as a result of the continuous continuation of the root $w_1(z)$ along the contour Λ_3 we return to the point z_0 with the value of the continued root, equal to the original root $w_1(z_0)$.

Therefore, in fact, the result of the continuation of the roots of the given equation along the various closed contours depends on the choice of the contour. As a result of such a continuation it is possible to obtain at a certain point z_0, starting out with one root of the equation $\Phi(w, z) = 0$, its remaining roots.

Example 2. Let us consider the algebraic function defined by the equation

$$\Phi = w^3-(z-b_1)^2(z-b_2) = 0.$$

Here, as also in the preceding example, b_1 and b_2 are certain complex constants. As a result of calculation we find, that in this case

$$D(z) = R(\Phi, \Phi'_w) = -27(z-b_1)^4(z-b_2)^2.$$

Thus, to the discriminantal set of the given equations there belong the points $z = b_1$ and $z = b_2$. Making the substitution $z = z^{-1}$, we

establish that the point $z_3 = \infty$ also belongs to the discriminantal set. Therefore, the domain $\mathscr{E}$ for the given equation is the complete plane, from which the points b_1, b_2, ∞ have been removed.

At a certain point $z \in \mathscr{E}$ the roots of the given equation are defined by the formulas†

$$w_1(z) = \sqrt[3]{(|z-b_1|^2\,|z-b_2|)}\; e^{i\frac{2\arg(z-b_1)+\arg(z-b_2)}{3}},$$

$$w_2(z) = w_1(z)\, e^{(2\pi i/3)}, \qquad w_3(z) = w_1(z)\, e^{(4\pi i/3)}.$$

As also in the preceding example, we shall consider the contours Λ_1, Λ_2 and Λ_3 and, also, the contour Λ_4 (Fig. 3).

The functions $w_1(z)$, $w_2(z)$ and $w_3(z)$ are continuous on the contour Λ_1 and hence after traversing it return to their original values.

Let us consider the contour Λ_2. In displacing the point z along the line $\Lambda_2^{(1)}$ from the point z_0 to the point z' the function $w_1(z)$ constitutes its own continuous continuation. At the point z' the function $w_1(z)$ suffers a discontinuity (here the index in the expression for $w_1(z)$ is decreased by $(4\pi/3)i$). To the values of the function $w_1(z)$ on the line $\Lambda_2^{(1)}$ there *continuously* follow on at the point z' the values of the function $w_2(z)$ on the line Λ_2. Also, on the line $\Lambda_2^{(2)}$ the function $w_2(z)$ is continuous, and hence, as a result of the continuous continuation of the root $w_1(z_0)$ along the contour we return to the point z_0 with the value of the continued root, equal to $w_2(z_0)$. It is possible to establish in exactly the same way, that as a result of the continuation along Λ_2 of the root $w_2(z_0)$ we arrive at the root $w_3(z_0)$, and as a result of the continuation of the root $w_3(z_0)$ we arrive at the root $w_1(z_0)$.

The continuous continuation of the roots $w_k(z_0)$ along the contour Λ_3 (on which the function $w_1(z)$ has two points of discontinuity) leads us once more to the original values of these roots.

The reader will easily verify himself, that the continuous continuation along the contour Λ_4 of the root $w_1(z_0)$ gives the root $w_3(z_0)$, of the root $w_2(z_0)$ gives the root $w_1(z_0)$, of the root $w_3(z_0)$ gives the root $w_2(z_0)$.

We once more see, that by means of analytic continuation it is possible to obtain at an arbitrary point z_0, starting with one root of the equation $\Phi(w_1\, z) = 0$, all its remaining roots.

Let us emphasize the fact, that in both examples the continuation along the contour Λ_3 returns the roots of our equations to their

† Our symbol $\sqrt[3]{\ }$ has its arithmetical meaning.

original values. Meanwhile both within and also outside this contour there are singular points of the algebraic functions considered. Therefore, for the roots to return to their original values after going round a certain contour it is sufficient for singular points of the algebraic function not to occur either inside or outside this contour, but this is by no means a necessary condition.

6. Singular points of an algebraic function

As before, we shall consider the algebraic function, defined by the equation

$$\Phi(w, z) = \phi_0(z)w^n+\phi_1(z)w^{n-1}+ \dots +\phi_n(z), \tag{1.44}$$

where $\phi_j(z)$ is a polynomial in z with highest degree m and $\phi_0(z) \neq 0$.

Definition. The point $z = \beta$ is said to be a multiple point of the algebraic function $w(z)$, defined by the equation $\Phi(w, z) = 0$, if the equation $\Phi(w, \beta) = 0$ has a root $w = \alpha$ of multiplicity $p > 1$.

Hence it follows, that at a multiple point $z = \beta$, if $\beta \neq \infty$, the equations

$$\Phi(w, \beta) = 0, \qquad \Phi'_w(w, \beta) = 0$$

have a solution in common. Consequently, β is a root of the discriminant $D(z)$ of the equation $\Phi(w, z)$. If $\beta = \infty$ is a multiple point of the function $w(z)$, then the equations

$$\Psi(w, 0) = 0, \qquad \Psi'_w(w, 0) = 0$$

must have a common solution. In this case $D_1(0) = 0$ (for the functions $\Psi(w, z)$ and $D_1(z_1)$, and their connexion with the functions $\Phi(w, z)$ and $D(z)$ see section 3 of the present chapter).

Thus, a multiple point of an algebraic function is always a singular point of it. On the other hand, it is obvious that the point $z = \beta$, at which the last relation holds, is always a multiple (and singular) point of the algebraic function $w(z)$. This means, that at the singular, but not multiple points of our algebraic function, we always have $\phi_0(\beta) = 0$ (correspondingly for $\beta = \infty$ the highest coefficient of the polynomial Ψ is equal to zero); this case is considered below. Let us note also, that we are now considering only the finite multiple roots of equation (1.44). The case of infinite multiple roots (and infinite values of an algebraic function in general) will be studied at the end of the present section.

An important part in the study of the function $w(z)$ in the neighbourhood of its multiple point is played by the following.

THEOREM 5. *If the equation* $\Phi(w, \beta)$, *where* $\beta \neq \infty$, *has a p-fold root* $w = \alpha$ *and* ϵ *is a certain sufficiently small positive number, then it is always possible to find a neighbourhood*† D *of the point* β *such that*:

$$(D): \qquad 0 < |z-\beta| < \rho, \tag{1.45}$$

at all the points $z \in D$ *there exist p and only p (simple, different) roots* $w = \omega_q(z)(q = 1, \ldots, p)$ *of the equation* $\Phi(w, z) = 0$, *satisfying the condition*

$$|w_q(z)-\alpha| < \epsilon. \tag{1.46}$$

Proof. In order to simplify the calculations which follow we shall introduce new variables. We will take the quantity

$$w-\alpha \tag{1.47}$$

as a new dependent variable, and the quantity

$$z-\beta \tag{1.48}$$

as a new independent variable. Thus, everything is reduced to the case, where $\alpha = \beta = 0$. We will preserve for the new variables the previous notation w and z. Therefore, we shall assume, that the given equation $\Phi(w, z)$ has at the point $z = 0$ a p-fold root $w = 0$. Our theorem will be proved, if we establish, that for the values of z, satisfying the inequality

$$(D_0): \qquad 0 < |z| < \rho, \tag{1.49}$$

the equation $\Phi(w, z) = 0$ has p and only p (simple, distinct) roots $\omega_q(z)$ where $q = 1, \ldots, p$ satisfying the condition

$$|\omega_q(z)| < \epsilon. \tag{1.50}$$

Passing to the proof of the last assertion, let us note, that thanks to our assumptions

$$\phi_{n-p+1}(0) = \ldots = \phi_n(0) = 0, \quad \phi_{n-p}(0) \neq 0. \tag{1.51}$$

In virtue of the last inequality and the continuity of the polynomial $\phi_{n-p}(z)$ at the point $z = 0$ there exist positive numbers ρ' and A, such that for $|z| < \rho'$ we have

$$|\phi_{n-p}(z)| > A. \tag{1.52}$$

† In the present and the following two sections the term neighbourhood of the point z has been given a different meaning from the ordinary one: the point z itself is considered as not belonging to its neighbourhood. As a result of this there is attached a different meaning to the expressions: "function regular at the point z", "function regular in the neighbourhood of the point z".

On the other hand, the modulus of a function, continuous in a closed region, is always bounded in that region, Hence there exists a number $B > 0$, such that for $|z| < \rho'$

$$|\phi_0(z)| < B, \ldots, |\phi_{n-p-1}(z)| < B. \tag{1.53}$$

Now let us put

$$\Phi(w, z) = \phi_{n-p}(z)w^p(1+P+Q), \tag{1.54}$$

where

$$\begin{aligned} P &= \frac{\phi_0}{\phi_{n-p}}w^{n-p}+\ldots+\frac{\phi_{n-p-1}}{\phi_{n-p}}w, \\ Q &= \frac{\phi_{n-p+1}}{\phi_{n-p}}\frac{1}{w}+\ldots+\frac{\phi_n}{\phi_{n-p}}\frac{1}{w^p}. \end{aligned} \tag{1.55}$$

For $|w| = r < 1$ by the inequalities (1.52) and (1.53)

$$\begin{aligned} |P| &\leqslant \left|\frac{\phi_0}{\phi_{n-p}}\right||w|^{n-p}+\ldots+\left|\frac{\phi_{n-p-1}}{\phi_{n-p}}\right||w| < \\ &< \frac{B}{A}(r+\ldots+r^{n-p}) < \frac{B}{A}(r+\ldots+r^{n-p}+r^{n-p+1}+\ldots) \\ &= \frac{B}{A}\frac{r}{r-1}. \end{aligned} \tag{1.56}$$

We will take

$$r = \epsilon < \frac{A}{A+2B}. \tag{1.57}$$

Then, as follows from the inequality (1.56),

$$|P| < \tfrac{1}{2}. \tag{1.58}$$

By virtue of the equations (1.51) and the continuity of the polynomials $\phi_{n-p+1}(z), \ldots, \phi_n(z)$ at the point $z = 0$ for every number $C > 0$ it is possible to find a number ρ (where $0 < \rho < \rho'$), such that when $|z| < \rho$,

$$|\phi_{n-p+1}(z)| < C, \ldots, |\phi_n(z)| < C. \tag{1.59}$$

Then for $|z| < \rho$, $|w| = \epsilon$

$$|Q| < \frac{C}{A}\left(\frac{1}{\epsilon^n}+\ldots+\frac{1}{\epsilon}\right) = \frac{C}{A}\frac{1-\epsilon^n}{\epsilon^n(1-\epsilon)}. \tag{1.60}$$

Let us take the number ϵ so small that we have

$$\frac{C}{A}\frac{1-\epsilon^n}{\epsilon^n(1-\epsilon)} < \tfrac{1}{2} \tag{1.61}$$

(thus, the value of the number ρ is determined by ϵ). Now, for $|z| < \rho$,

$$|Q| < \tfrac{1}{2}. \tag{1.62}$$

As $\rho < \rho'$, then in the circle $|z| < \rho$ the inequality (1.58) (for $|w| = \epsilon$) holds.

Let us fix z at a certain value z_0 in the circle $|z| < \rho$ and displace the point w along the circumference $|w| = \epsilon$ in a direction, corresponding to an increase of the argument of w. We shall consider the expression (1.54) for $\Phi(w, z_0)$. As a result of describing the circumference $|w| = \epsilon$ and returning the point w to its original position the argument of w increases by 2π, the argument of w^p by the quantity $2\pi p$. The complex number $U = 1+P+Q$ describes during this a certain closed curve on the complex plane. This curve, thanks to the inequalities (1.58) and (1.62), lies entirely within the circle $|U-1| < 1$. In fact, we have

$$|U-1| = |(1+P+Q)-1| = |P+Q| < |P|+|Q| < 1.$$

As in the circle $|U-1| < 1$ all the continuous branches of the argument of U are single-valued, the values which the argument of the complex number $U = 1+P+Q$ acquires on the return of the point w to its original position on the circumference $|w| = \epsilon$, is identical with its original value; the increment of the argument is equal to zero.

Therefore, the total change of the argument of the quantity $\Phi(w, z_0)$ on describing the circumference $|w| = \epsilon$ is equal to $2\pi p$. Hence, using "the principle of the argument",† we conclude that the equation $\Phi(w, z_0) = 0$ has p roots (counting each multiple root as many times as its multiplicity).

At the point $z = 0$ the discriminant of the polynomial $\Phi(w, z) = 0$ is equal to zero (as the equation $\Phi(w, 0) = 0$ has a multiple root). Hence, if the number ρ is sufficiently small, then in the circle $|z| < \rho$ there is no other zero of this discriminant, apart from the point $z = 0$. Then the domain $D_0 \subset \mathscr{E}$, and consequently, the equation $\Phi(w, z_0)$ has only simple roots.

Thus, theorem 5 is completely proved.

† See F.C.V., Chap. VII, Art. 77, formula 39.

Let us note (we will use the previous notations) that, in particular, it is true also when α is a simple root of the equation $\Phi(w, \beta) = 0$. In this case, the assertion of the theorem about the existence of a root of the equation $\Phi(w, z) = 0$, $(z \in D_0)$, close to α, that is a root, satisfying the condition (1.46), also follows from theorem 3, insofar as the latter establishes, that for $\Phi'_n(\alpha, \beta)$ there exists a branch $w_0(z)$ of our algebraic function, regular in the neighbourhood of the point $z = \beta$ for which $w_0(\beta) = \alpha$.

The fact, that at the points $z \in D$ there is under the given conditions only one root of the equation $\Phi(w, z) = 0$, satisfying the inequality (1.46), also follows from theorem 3, if the point $\beta \in \mathscr{E}$: then as $z \to \beta$ the other roots of the equation $\Phi(w, z) = 0$ tend to other roots of the equation $\Phi(w, \beta) = 0$ (as they are the values of other regular branches of the algebraic function $w(z)$). In the contrary case, if the point β does not belong to the domain $\mathscr{E}$, theorem 5 is necessary for our result.

Let us turn to the consideration of the root $w = \alpha$ of the arbitrary multiplicity p. In this case the equation $\Phi(w, z) = 0$ possesses at the points $z \in D$, p roots, near to α (satisfying the inequality (1.46)). If we denote these roots by $\omega_1, \ldots, \omega_p$, and by $\omega_{p+1}, \ldots, \omega_n$ the remaining roots of the equation $\Phi(w, z) = 0$, then there exist numbers $\epsilon_1 > 0$ and $\epsilon_2 > 0$, where $\epsilon_1 < \epsilon_2$ such that at all the points $(z \in D$

$$|\omega_s - \alpha| < \epsilon_1, \quad |\omega_t - \alpha| > \epsilon_2 \tag{1.63}$$

here $s = 1, \ldots, p$; $t = p+1, \ldots, n$).†

Now let there be defined at all the points z of the domain D some continuous function $f(z)$, which assume there values, which are necessarily roots of the equation $\Phi(w, z) = 0$. Then if at some point $z' \in D$ this function is equal to some root $\omega_s(z')$, where $s = 1, \ldots, p$, then at another point $z'' \in D$ it cannot be equal to one of the roots $\omega_t(z'')$, where $t = p+1, \ldots, n$. In fact, if this is not so the function $f(z)$ would assume on every line $L \subset D$, connecting the points z' and z'', values continuously connecting the quantities $\omega_s(z')$ and $\omega_t(z'')$ and being roots of the equation $\Phi(w, z) = 0$ for all $z \in L$. This is impossible because of the relations (1.63).

† This happens because as $z \to \beta$ the roots $\omega_{p+1}, \ldots, \omega_n$ approach other roots of the equation $\Phi(w, \beta) = 0$ or their moduli increase without limits. For this last possibility see the end of the present article.

Let us now take some point $z_0 \in D$ and in its neighbourhood determine the n regular branches of our algebraic function. We shall continue analytically these functions into the domain $\tilde{D}$, obtained from the circle D by removing from it the radius R, lying on the continuation of the segment $z_0\beta$ (Fig. 4). The functions $w_1(z), \ldots,$ $w_n(z)$ determined in this way will be regular (in particular, single-valued) in the domain $\tilde{D}$, as this domain is simply connected and constitutes part of the domain $\mathscr{E}$.† On the line R the functions $w_k(z)$ may suffer a discontinuity. If the latter occurs and, for example, the values of the functions $w_1(z)$ on one side of the radius R do not constitute a continuous continuation of the values of this function on the other side, then such a continuous continuation is

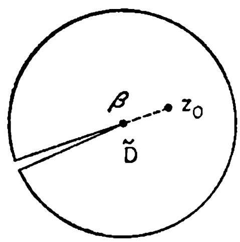

FIG. 4

given by the values of some other one of the functions $w_k(z)$. We arrive at this result because in the neighbourhood of every point of the radius R (apart from the point β, they belong to the domain $\mathscr{E}$) the roots of the equation $\Phi(w, z) = 0$ form n regular (consequently, continuous) functions which are the branches of our algebraic function.

We shall suppose, that in fact the quantities $w_1(z_0), \ldots, w_p(z_0)$ satisfy the first of the inequalities (1.63). Let the function $w_1(z)$ be continued analytically along the circumference L of the circle with centre at the point β, passing through the point z_0 (this circumference can be replaced by any other contour, lying in the domain D and

† It is useful to emphasize the distinction between the functions $w_k(z)$ and $\omega_k(z)$. Both the former and the latter are branches of the algebraic function $w(z)$; however, the $w_k(z)$ are regular functions, but the $\omega_k(z)$ are arbitrary branches of $w(z)$, among which somehow at every point z (independently of the values of these branches at other points) its values are distributed. In this only the necessity of conforming to the equations (1.63) is taken into account.

going once round the point β). This analytic continuation reduces to a continuous continuation along the line L (here it is necessary only to take care, that the continued function must have at every point $\zeta \in L$ a value, which is a root of the equation $\Phi(w, \zeta) = 0$). As we have just pointed out, at the point of intersection of the circumference L with the radius R the function $w_1(z)$ may have a discontinuity. Beginning with this point it is possible, that we may obtain as a continuous continuation the value of some other branch $w_k(z)$. As the function to be continued let us take the function $f(z)$ of our previous discussion. We shall see, that as the indicated branch $w_k(z)$ we can obtain only one of the functions

$$w_1(z), \ldots, w_p(z). \tag{1.64}$$

If it is found that $w_k(z) = w_1(z)$, then the latter function will be single valued, and consequently, also regular in the domain D. Thanks to its continuity at the point β (which follows from theorem 5) it is regular also at the same point β.† In this case we shall say, that the function $w_1(z)$ constitutes a separate *cyclical system* of branches of the algebraic function in the neighbourhood of the point β.

Let us assume now, that $w_k(z) \neq w_1(z)$. In case of necessity changing the numbering of the functions (1.64), we can always take $w_k(z) = w_2(z)$. Let the branch $w_2(z)$ be continued along the circumference L, traversing it in the previous direction. On a repeated crossing of the radius R the branch $w_2(z)$ cannot again give the continuous continuation of the function considered, (as its value across the radius R continuously follows on to the values up to the radius R of the branch $w_1(z)$, and not of $w_2(z)$). If as such a continuation we obtain the function $w_1(z)$, then the branches $w_1(z)$ and $w_2(z)$ form a cyclical system.‡ If, however, as the continuous continuation of the branch $w_2(z)$ across the radius R there is a new function of the system (1.64), we take the latter as $w_3(z)$ and once more continue it along the circumference L and so on. As the number of functions, which can be obtained as a result of such a continuation, is finite (it is equal to p), after a certain number ν of intersections of

† On this subject see F.C.V., Chap. VI, Art. 63.

‡ In this case the values of the functions $w_1(z)$ and $w_2(z)$ on the line join one another like the values of the branches $\sqrt{z}$ on the positive real semiaxis (see F.C.V., Chap. III, Art. 25, and Fig. 29).

the radius R we shall return to the original function $w_1(z)$. In this case we shall say, that the functions

$$w_1(z), \ldots, w_\nu(z) \tag{1.65}$$

constitute a *cyclical system of branches of the algebraic function* $w(z)$ (in the neighbourhood of the given singular point).

If $\nu < p$, we then take some one of the functions $w_k(z)$, where $k = \nu+1, \ldots, p$ and take it as $w_{\nu+1}(z)$. We once more continue it along the circumference L. Repeating the previous argument, we determine a second cyclical system of branches of the algebraic function

$$w_{\nu+1}(z), \ldots, w_{\nu+\mu}(z). \tag{1.66}$$

Finally all the functions $w_1(z), \ldots, w_p(z)$ will be sorted out into some such cyclical systems.

The functions of each cyclical system pass consecutively into one another according to the cyclical law, when z goes round the multiple point β along the circumference L (or some other contour, going once round the point β and lying inside the domain D). Hence every cyclical system of branches defines in the domain D an analytic function.† The number of values assumed by this analytic function at the point $z \in D$, is equal to the number of regular branches, comprising the cyclical system considered. In a particular case, if some regular branch comprises a separate cyclical system, the corresponding analytic function will be single valued and identical with this regular branch.

We will denote the analytic function, determined by the first cyclical system $w_1(z), \ldots, w_\nu(z)$ by the symbol $w_{\mathrm{I}}(z)$, by the second cyclical system $w_{\nu+1}(z), \ldots, w_{\nu+\mu}(z)$ by the symbol $w_{\mathrm{II}}(z)$ and so on. These functions $w_{\mathrm{I}}(z)$, $w_{\mathrm{II}}(z), \ldots$, generally speaking, are many-valued branches of the algebraic function $w(z)$.

If the number of regular branches $w_q(z)$, forming such an analytic function, is ρ and $\rho > 1$, then the point β is said to be a *critical point of it of multiplicity* ρ (or a *branch point of multiplicity* ρ). In this case the point β is considered also to be a critical point of multiplicity ρ of the whole algebraic function $w(z)$.

Thus, at a multiple point β of an algebraic function $w(z)$, generally speaking, there are located several of its critical points.

† See F.C.V., Chap. VI, Art. 63.

If $\rho = 1$, the corresponding analytic function ($w_{\mathrm{I}}(z)$, or $w_{\mathrm{II}}(z)$, or $w_{\mathrm{III}}(z)$ and so on) is regular at the point β.

Let us note, that in a particular case the branches $w_1(z), \ldots, w_p(z)$ may comprise p cyclical systems. Then each of them will contain only one function; all these functions will be regular at the point β. After going round this point they will all return to the original value (we encountered this fact in the examples, considered at the end of the preceding article). In this case we shall consider, that at the multiple point β there are located p points of regularity of the branches of the algebraic function, which assume there identical values.

The case $\beta = \infty$ does not introduce any material changes in the results of the preceding investigations. We make a change of variable, putting $z = z_1^{-1}$, and consider the equation $\Psi(w, z_1) = 0$ in the neighbourhood of the point $z_1 = 0$. Then by means of the converse substitution we shall obtain p regular (in the neighbourhood of the point $z = \infty$) branches of our algebraic function, tending to α as $z \to \infty$, arranged in several cyclical systems. The functions of each system pass into one another, when the point z goes round the point at infinity along a contour sufficiently close to it.

We will now turn to the consideration of poles of an algebraic function.

Definition. The point $z = \beta$ is said to be a *pole* of the algebraic function $w(z)$, defined by the equation $\Phi(w, z) = 0$, if for every number $M > 0$ it is possible to find a neighbourhood of the point β such that at every point of this neighbourhood the equation $\Phi(w, z) = 0$ has a root ω for which

$$|\omega| > M. \tag{1.67}$$

In this case we shall say for brevity, that the equation $\Phi(w, z) = 0$ has at the point β the root $w = \infty$.

Let the point β be a pole of the algebraic function $\Phi(w, z) = 0$. Let us put $w = W^{-1}$ and consider the equation

$$\begin{aligned} \chi(W, z) &= W^n\Phi(1/W, z) = \\ &= \phi_0(z)+\phi_1(z)W+ \ldots +\phi_n(z)W^n = 0. \end{aligned} \tag{1.68}$$

This equation has in the neighbourhood of the point $z = \beta$ roots as close to zero as desired. It cannot have roots identically equal to zero, as $\phi_0(z) \neq 0$. Hence, using theorem 5, it is easy to establish

that the equation $\chi(W, B) = 0$ has the root $W = 0$. Hence in our case $\phi_0(\beta) = 0$, and consequently, also $D(\beta) = 0$, where $D(z)$ is the discriminant of the equation $\Phi(w, z) = 0$. Therefore, the poles of an algebraic function form a variety of its singular points.

Conversely, if $\phi_0(\beta) = 0$, but not all the $\phi_k(\beta) = 0$, the point β is a pole of the algebraic function, defined by the equation $\Phi(w, z) = 0$. In fact, the equation $\chi(W, \beta)$ then has the root $W = 0$, the equation $\chi(w, z) = 0$ in the neighbourhood of the point $z = \beta$ has roots (by theorem 5), as close to zero as desired, and the equation $\Phi(w, z)$ has roots, given in this neighbourhood by the condition (1.68). Thus, the point β is in fact a pole of our algebraic function.

The singular points of the algebriac function $w(z)$ are determined as the roots of the equation $D(z) = 0$. At these points $z = \beta$ either the equations

$$\Phi(w, \beta) = 0, \qquad \Phi'_w(w, \beta) = 0, \tag{1.69}$$

can be satisfied by the same value of w or it is necessary that ${}_0\phi(\beta) = 0$. In the first case $\Phi(w, \beta) = 0$ has a multiple root, and the point $z = \beta$ is a multiple point, in the second case the point $z = \beta$ is a pole. The case, where all the $\phi_k(\beta) = 0$, is considered below, it does not lead to a new kind of singular point.†

Thus, *multiple points and poles are the only kind of singular points which algebraic functions can have.*

Let us turn to the consideration of the poles of an algebraic function. If at the point $z = \beta$

$$\phi_0(\beta) = 0, \quad \phi_1(\beta) = 0, \ldots, \phi_{p-1}(\beta) = 0, \quad \phi_p(\beta) \neq 0, \tag{1.70}$$

then the equation $\chi(W, \beta) = 0$ will have a root $W = 0$ of multiplicity p, and its remaining roots will be different from zero. Corresponding to this the equation $\Phi(w, \beta)$ will be an equation of degree $n-p$ and will have $n-p$ finite roots (not necessarily distinct).

In this case we shall say, that the equation $\Phi(w, z) = 0$ has at the point $z = \beta$ a p-fold root $w = \infty$, and the point β is a p-fold *pole*‡ *of the algebraic function* $w(z)$. In the neighbourhood of such a point there are defined p regular branches of the function $w(z)$, the moduli of which increase without limit as $z \to \beta$.

These branches, just as before, comprise one or several cyclical systems. Every such system determines in the neighbourhood of

† See the last paragraph of the present section.

‡ This multiplicity should not be confused with the order of the pole of a single-valued function.

the point β an analytic function, which is a (generally speaking, many-valued) branch of the algebraic function $w(z)$.

If $\rho > 1$ is the number of regular branches $w_q(z)$, forming the given analytic function, then the point β is said to be a *critical pole of this many-valued branch of multiplicity* ρ. In this case the point β is also considered to be a critical pole of multiplicity ρ of the whole algebraic function $w(z)$.

If $\rho = 1$, then the corresponding analytic function is regular in the neighbourhood of the point β and has the latter as a pole of single-valued character (later on we shall speak of "poles of multiplicity one").

It is obvious, that at a certain point β there may be located several singularities of an algebraic function (belonging to the various branches of it, which are regular in the neighbourhood of the point β).

The case $\beta = \infty$ reduces to the preceding with the help of the substitution $z = z_1^{-1}$.

Let us also consider the case, where all the coefficients $\phi_0(z), \ldots, \phi_n(z)$ of the equation $\Phi(w, z) = 0$ become zero at the point β. Then these coefficients, and with them also the entire polynomial $\Phi(w, z)$, are divisible by a certain power of the difference $z-\beta$. At $z = \beta$ the equation $\Phi(w, z) = 0$ is satisfied by any value of w. Dividing this equation by the above mentioned power of the difference $z-\beta$, we arrive at a new equation $\Phi_1(w, \beta) = 0$, for which β is a point of one of the types considered above; at the point β the roots of this equation have a well determined (finite or infinite) value. These roots are the limits of the roots of the equation $\Phi(w, z) = 0$ as $z \to \beta$. Thus, it is clear, that at the point β the algebraic function $w(z)$ has a *removable* singularity. In what follows we shall always define the function $w(z)$ at such a point, by taking as its value there the root of the equation $\Phi_1(w, z) = 0$ at $z = \beta$. Let us note, that such a convention is equivalent to the replacement of the equation $\Phi(w, z) = 0$ by the equation $\Phi_1(w, z) = 0$ throughout the whole of the z-plane.

7. The representation of the branches of an algebraic function by series in the neighbourhood of its singular points

Initially we will consider the case of a finite multiple point $z = \beta$ of the algebraic function $w(z)$, defined by equation (1.44). Let the functions

$$w_1(z), \ldots, w_\nu(z) \tag{1.71}$$

tend to α as $z \to \beta$, be regular in the domain $\tilde{D}$ and comprise in the domain D a cyclical system of branches of the function $w(z)$. Here and later on we attach to all the notations the same meaning, as in the preceding article; however we also agree to locate the point z_0 in the domain D so that the difference $z_0-\beta$ is a positive number, and in the determination of the branches (1.71) to go round the circumference L in the direction of increase of the argument of the quantity $z-\beta$.†

Let us introduce a new independent variable, putting

$$z-\beta = \zeta^{\nu}. \tag{1.72}$$

In the mapping (1.72) to the domain $\tilde{D}$ there corresponds in the ζ-plane ν pre-images $\Delta_1, \dots, \Delta_\nu$, occupying (Fig. 5) the circle with

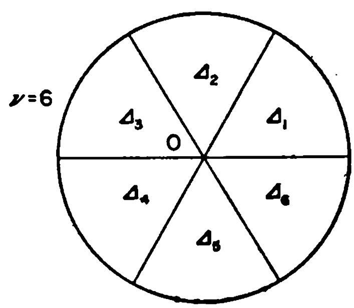

FIG. 5

centre at the point $\zeta = 0$ of radius $\rho^{1/\nu}$ (the radii $r_1, \dots, r_\nu$ going away from this circle are the pre-images of the radius R of the circle D). We will determine at the points ζ of each of the domains Δ_k the function $w_k(\beta+\zeta^\nu) = w_k(z)$. In their totality they determine in the circle $|\zeta| < \rho^{1/\nu}$ a single-valued and differentiable, that is, regular function, which we will denote by the symbol $w_{\mathrm{I}}(\zeta)$. On each of the radii r_k (where $k = 1, \dots, \nu-1$) we ascribe to this function the value, to which the functions $w_k(\beta+\zeta^\nu)$ and $w_{k+1}(\beta+\zeta^\nu)$,

† The circumference L can be replaced by any contour, lying in the domain D and going once round the point β. If in going round the circumference L in the direction indicated in the text the circle bounded by it remains on the left (this is the case if, as is usually done, we use a right hand co-ordinate system), then the contour mentioned must also be traversed so that the domain bounded by it remains on the left of the direction of traversal. In the contrary case this direction must be replaced by the opposite one.

prescribed in the domains Δ_k and Δ_{k+1}, separated from one another by the radius r_k, tend, as the point ζ approaches that radius. On the radius r_ν we ascribe to the function $w_{\mathrm{I}}(\zeta)$ the value, to which the functions $w_\nu(\beta+\zeta^\nu)$ and $w_1(\beta+\zeta^\nu)$ tend as the point ζ approaches that radius.

It is then possible to expand in integral powers of ζ the function $w_{\mathrm{I}}(\zeta)$ which is regular in the circle $|\zeta| < \rho^{1/\nu}$. Taking into consideration also the equation $w_{\mathrm{I}}(0) = \alpha$, we find, that for $|\zeta| < \rho^{1/\nu}$

$$w_I(\zeta) = \alpha+a_1\zeta+a_2\zeta^2+\dots \tag{1.73}$$

Let us now return to the previous independent variable z. We have

$$\begin{aligned}\zeta &= (z-\beta)^{1/\nu}\\ &= {}^{\nu}(|z-\beta|)\, e^{i[\arg(z-\beta)+2k\pi]/\nu} = (z-\beta)^{1/\nu}{}_{\mathrm{PR}}\omega^{k-1},\end{aligned} \tag{1.74}$$

where $k = 1, \dots, \nu$.

$$(z-\beta)^{1/\nu}{}_{\mathrm{PR}} = {}^{\nu}(|z-\beta|)\, e^{i[\arg(z-\beta)]/\nu}, \quad \omega = e^{2\pi i/\nu}. \tag{1.75}$$

In the formulas (1.74) and (1.75) the symbol $^\nu$ has its arithmetical meaning.

Let z be a certain point of the circle $|z-\beta| < \rho$. Then by the definition of the function $w_{\mathrm{I}}(\zeta)$, the sum of the series (1.73) gives us at the point ζ, indicated by formula (1.74), the value of the function $w_k(\beta+\zeta^\nu) = w_k(z)$ at this point z. Thus we find, that

$$\begin{aligned} w_1(z) &= \alpha+a_1(z-\beta)^{1/\nu}{}_{\mathrm{PR}}+a_2(z-\beta)^{2/\nu}{}_{\mathrm{PR}}+\dots,\\ w_2(z) &= \alpha+a_1(z-\beta)^{1/\nu}{}_{\mathrm{PR}}\omega+a_2(z-\beta)^{1/\nu}{}_{\mathrm{PR}}\omega^2+\dots,\\ &\dots\dots\dots\dots\dots\dots\dots\dots \end{aligned} \tag{1.76}$$

and in general

$$w_k(z) = \alpha+a_1(z-\beta)^{1/\nu}{}_{\mathrm{PR}}\omega^{k-1}+a_2(z-\beta)^{2/\nu}{}_{\mathrm{PR}}\omega^{2(k-1)}+\dots \tag{1.77}$$

Therefore, the branches $w^\nu(z), \dots, w^\nu(z)$ of the algebraic function $w(z)$, which tend to one and the same quantity α as $z \to \beta$ and forming in the neighbourhood of the point $z = \beta$ a cyclical system, are expanded into the series (1.76).

Putting $\zeta = (z-\beta)^{1/\nu}$ into the expression for the function $w_{\mathrm{I}}(\zeta)$ we determine in the domain D a ν-valued function, Its ν values corresponding to a certain point $z \in D$, are the values of the functions $w_k(z)$ $(k = 1, \dots, \nu)$. Consequently, the ν-valued function

lefined by us in the domain D is identical with the analytic function $w_{\mathrm{I}}(z)$ considered above (in the preceding article).

From equation (1.73) it follows that in the neighbourhood of the point β the function $w_{\mathrm{I}}(z)$ is represented by the series

$$w_{\mathrm{I}}(z) = \alpha + a_1(z-\beta)^{1/\nu} + a_2(z-\beta)^{2/\nu} + \dots \qquad (1.78)$$

If in the neighbourhood of the multiple point β the branches of the algebraic function form several cyclical systems, we, as before, will represent them by series of the form (1.77). These cyclical systems define in the domain D the analytic functions $w_{\mathrm{II}}(z)$, $w_{\mathrm{III}}(z)$, and so on. The latter are represented by series of the type (1.78).

Those of the analytic functions $w_{\mathrm{I}}(z)$, $w_{\mathrm{II}}(z)$, ..., which correspond to cyclical systems, comprising more than one regular branch, are many-valued. Then the corresponding series of type (1.78) contains fractional powers of $z-\beta$.

We shall now study the behaviour of the algebraic function $w(z)$ in the neighbourhood of a critical point at infinity. In this case the algebraic function $w(z)$, defined by the equation $\Psi(w, z_1) = 0$ (see formula (1.16)), has the critical point $z_1 = 0$. Let the functions

$$w_1(z_1), \dots, w_\nu(z_1) \qquad (1.79)$$

form a cyclical system of branches of the function $w_1(z)$ in the neighbourhood of the point $z_1 = 0$, tending to the value α as $z_1 \to 0$. Together they form in this neighbourhood (excluding the point $z_1 = 0$) a ν-valued analytic function $w_{\mathrm{I}}(z_1)$. The latter is represented for $|z_1| < \rho$ by the series

$$w_{\mathrm{I}}(z_1) = \alpha + a_1 z_1^{1/\nu} + a_2 z_1^{2/\nu} + \dots \qquad (1.80)$$

As a result of the substitution $z_1 = z^{-1}$ we return to the algebraic function $w(z)$, and in place of the series (1.80) we obtain the series

$$w_{\mathrm{I}}(z_1) = \alpha + a_1 z^{-1/\nu} + a_2 z^{-2/\nu} + \dots, \qquad (1.81)$$

which converges in the neighbourhood of the point $z = \infty$ to the corresponding function $w_{\mathrm{I}}(z)$; the function $w_{\mathrm{I}}(z)$ is a ν-valued, analytic function in the neighbourhood of the point $z = \infty$, having if $\nu > 1$ the latter as a critical point of multiplicity ν.

Now let us suppose, that the point β is a pole of multiplicity one of the algebraic function $w(z)$. By means of the substitution $w = W^{-1}$ we pass to the equation $\chi(W, \beta) = 0$ (see formula (1.68)). The equation $\chi(W, \beta) = 0$ will have a simple root $W = 0$. In the neighbourhood of the point β there will exist a regular branch $W_1(z)$ of

the algebraic function $W(z)$, defined by the equation $\chi(W, z) = 0$, which becomes zero at $z = \beta$. Let the point β be a zero of order l of the function $W_1(z)$. Then in the neighbourhood of the point β

$$W_1(z) = (z-\beta)^l[b_0+b_1(z-\beta)+ \dots], \tag{1.82}$$

where $b_0 \neq 0$. Thus, we find,† that the corresponding branch $w_1(z)$ of the algebraic function $w(z)$ is represented in the neighbourhood of a pole of the function (the point $z = \beta$) by the series

$$w_1(z) = \frac{a_{-l}}{(z-\beta)^l}+ \dots + \frac{a_{-1}}{z-\beta}+a_0+a_1(z-\beta)+ \dots \tag{1.83}$$

The number l is the order of this pole.

Let us consider also the case, where the point $z = \infty$ is a pole of multiplicity one of the algebraic function. It is easy to see, that then instead of the expansion (1.83) we shall obtain in the neighbourhood of the point at infinity the series

$$w_1(z) = a_{-l}z^l+ \dots +a_{-1}z+a_0+a_1/z+ \dots \tag{1.84}$$

Let us now suppose, that the point β (where $\beta \neq \infty$) is a critical pole of multiplicity ν of the algebraic function $w(z)$. By means of the substitution $w = W^{-1}$ we once more pass to the equation $\chi(W, z) = 0$. The equation $\chi(W, \beta) = 0$ will, subject to our conditions have a root $W = 0$ of a certain multiplicity $p \geqslant \nu$. Let the functions

$$W_1(z), \dots, W_\nu(z) \tag{1.85}$$

tend to zero as $z \to \beta$ and constitute in the neighbourhood of this point a cyclical system belonging to the algebraic function $W(z)$. We, as before, will form the function $W_{\mathrm{I}}(\zeta)$ regular at the point $\zeta = 0$ (as before we have $z-\beta = \zeta^\nu$). By our conditions $W_{\mathrm{I}}(0) = 0$. Let the point $\zeta = 0$ be for this function a zero of multiplicity λ. Then in the neighbourhood of the point $\zeta = 0$ it is represented (instead of (1.73)) by the series

$$W_{\mathrm{I}}(\zeta) = \zeta^\lambda(b_0+b_1\zeta+ \dots), \tag{1.86}$$

where $b_0 \neq 0$. Now let us put $\zeta = (z-\beta)^{1/\nu}$. We shall obtain in the domain D the analytic ν-valued function

$$W_{\mathrm{I}}(z) = (z-\beta)^{\lambda/\nu}[b_0+b_1(z-\beta)^{1/\nu}+b_2(z-\beta)^{2/\nu}+ \dots]. \tag{1.87}$$

All the ν values of this function tend to zero, when $z \to \beta$.

† See F.C.V., Chap. VI, Art. 67.

We have to return to the original function $w(z)$. For this it is necessary to perform the converse substitution $W = w^{-1}$. As a result we determine in the domain D the ν-valued analytic function

$$w_{\mathrm{I}}(z) = (z-\beta)^{-\lambda/\nu}[a_0+a_1(z-\beta)^{1/\nu}+a_2(z-\beta)^{2/\nu}+\ \dots\ ]. \tag{1.88}$$

The branches $w_1(z), \dots, w_\nu(z)$ of this function regular in the domain $\tilde{D}$ form there a cyclical system of branches of the algebraic function $w(z)$. All the functions $w_1(z), \dots, w_\nu(z)$ tend to infinity as $z \to \beta$.

Finally, we have to consider the case, where the point $z = \infty$ is a critical pole. Repeating the previous reasoning, we find, that the branches of the algebraic function $w(z)$, which form a cyclical system in the neighbourhood of the point $z = \infty$ and tend to infinity as $z \to \infty$ turn out to be the regular branches of the ν-valued analytic function $w_{\mathrm{I}}(z)$. The latter is represented in the neighbourhood indicated by the series

$$w_{\mathrm{I}}(z) = z^{\lambda/\nu}(a_0+a_1z^{-1/\nu}+a_2z^{-2/\nu}+\ \dots\ ) \tag{1.89}$$

Let us summarize the results of our investigation in the following proposition:

THEOREM 6. *The branches of the algebraic function $w(z)$, which comprise a cyclical system in the neighbourhood of its critical point β, in their totality constitute an analytic function. The latter is given by the series* (1.78), *if $\beta \neq \infty$, or by the series* (1.81), *if $\beta = \infty$.*

In the neighbourhood of a pole β of multiplicity one, of the algebraic function one of its branches is represented by the series (1.83), *if $\beta \neq \infty$, or by the series* (1.84), *if $\beta = \infty$.*

The branches of the algebraic function $w(z)$, constituting a cyclical system in the neighbourhood of its critical point β, in their totality constitute an analytic function. The latter is given by the series (1.88), *if $\beta \neq \infty$, or by the series* (1.89), *if $\beta = \infty$.*

Newton's diagram

We will now indicate methods for the practical construction of the series, considered in the preceding article. Let β be a multiple point of the algebraic function $w(z)$, defined by the equation $\Phi(w, z) = 0$ (see formula (1.44)): we shall assume, that the equation $\Phi(w, \beta) = 0$ has a p-fold root $w = \alpha$. Let us take the quantity $w-\alpha$ as the new dependent variable and the quantity $z-\beta$ as the new independent variable. The general case then reduces to the case, where $\alpha = \beta = 0$. We will retain for the new variables the previous notations w and z.

Therefore, we shall suppose, that the equation $\Phi(w, 0) = 0$ has a p-fold root $w = 0$. Then (see the relation (1.53)),

$$\phi_{n-p+1}(0) = \ldots = \phi_n(0), \qquad \phi_{n-p}(0) \neq 0.$$

Let us put

$$\phi_{n-k}(z) = a_{k,m}z^m + \ldots + a_{k,0} \quad (k = 0, 1, \ldots, p). \tag{1.90}$$

Thanks to the relations (1.53)

$$a_{p-1,0} = a_{p-2,0} = \ldots = a_{0,0} = 0, \quad a_{p,0} \neq 0. \tag{1.91}$$

Let s_k be the lowest power of z in the polynomial $\phi_{n-k}(z)$. Here the subscript k assumes those values (from the $p+1$ possible values $0, 1, \ldots, p$) for which $\phi_{n-k}(z) \not\equiv 0$. Let us note, that we have $\phi_{n-p}(z) \not\equiv 0$. In addition to this, we can always suppose that $\phi_n(z) \not\equiv 0$. If this were not so the equation $\Phi(w, z) = 0$ would have the root $w = 0$ of some multiplicity $p_1 \leqslant p$. If $p_1 < p$ (it is only necessary to study this case, as for the root $w = 0$ all the coefficients of the series (1.78) are known; they are equal to zero), then in order to determine the remaining $p-p_1$ roots of the equation $\Phi(w, z) = 0$, which tend to zero as $z \to 0$, it is necessary to divide the latter by w^{p_1} and then consider the equation with non-zero, absolute (with respect to w) term.

We wish to find series of the form

$$a_1 z^{\mu_1} + a_2 z^{\mu_2} + \ldots, \tag{1.92}$$

satisfying the equation $\Phi(w, z) = 0$. Here $a_1 \neq 0$, $a_2 \neq 0$, and so on, $\mu_1 < \mu_2 < \ldots$ are certain rational numbers. We take the series (1.92) instead of the series (1.76) or the series (1.78), as some of the terms of these series may turn out to be equal to zero.

Let us substitute the series (1.92) in equation (1.44) in place of w. Let

$$F(z) = \Phi(a_1 z^{\mu_1} + a_2 z^{\mu_2} + \ldots, z). \tag{1.93}$$

As a result of the substitution the equation $\Phi(w, z) = 0$ must become an identity, that is after the collection of similar terms all the coefficients of the series $F(z)$ must become equal to zero. Thus, we obtain relations for the determination of the unknown coefficients of the series (1.92).

So long as we do not know what μ_1 is equal to, we cannot say which in fact of the terms of the series $F(z)$ will be the least, that is, will contain the smallest power of z. It is possible only to assert

that they will originate from the substitution of the series (1.92) into the monomial

$$a_{k,s_k} w^k z^{s_k} \tag{1.94}$$

(where the index k by the above, assumes those values from $0, 1, \ldots, p$ (which are the possible ones) for which $\phi_{n-k}(z) \not\equiv 0$) and hence will be one among the expressions

$$a_{k,s_k} z^{s_k+\mu_1 k}. \tag{1.95}$$

In fact, the degree of the term $a_1{}^p a_{p,0} z^{\mu_1 p}$ (let us remember that our $a_{p,0} \neq 0$) is less than the degree of any other component of $F(z)$, obtained from the substitution of the series (1.92) into the section $\phi_0(z)w^n + \ldots + \phi_{n-p}(z)w^{n-p}$ of the polynomial $\Phi(w, z)$, and the degree of the term $a_1{}^k a_{k,s_k} z$ is less that the degree of any other component of $F(z)$, obtained from $\phi_{n-k}(z)w^k$ for $k = p-1, \ldots, 0$.

Let us note also that, as $a_1 \neq 0$, then among the expressions (1.95), at least two must have this least degree. The latter circumstance forms the basis of an extremely simple method, due to Newton, which enables all the possible values of the number μ_1 to be found.

Let us construct on some auxiliary plane the points with co-ordinates s_k *and* k (*the index assumes the previous values*). On Fig. 6 the net of these points has been constructed for the case, where $p = 6$ (and hence $s_6 = 0$), $s_5 = 4$, $s_4 = 2$, $s_2 = 3$, $s_1 = 5$, $s_0 = 8$, and $\phi_{n-3}(z) \equiv 0$. Let us draw through all the points (s_k, k) of our net the straight lines

$$y + \rho x = \text{const.}, \tag{1.96}$$

forming with the ordinate axis a certain fixed angle ϕ (we have $\tan \phi = \rho$). The distance from the co-ordinate origin to such a straight line is equal to†

$$\frac{s_k + k\rho}{\sqrt{(1+\rho^2)}},$$

that is, differs by the factor $1/\sqrt{(1+\rho^2)}$ (the same for all these straight lines) from the quantity $s_k + \rho k$—the index of the lowest term, obtained from the substitution of the series (1.92) (where $\mu_1 = \rho$) in the monomial (1.94).

† This follows from a well known formula of plane analytic geometry.

Thus the lowest term of the series $F(z)$ (where $\mu_1 = \rho$) is obtained from that monomial of (1.94), for which the straight line which we have constructed is *nearest* to the co-ordinate origin. If this straight line passes through only one point (s_k, k) of our net, then the lowest

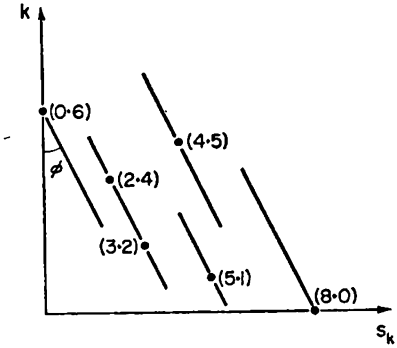

FIG. 6

degree in $F(z)$ will have only one component of the form (1.95). Hence (as $F(z) \equiv 0$) it inevitably follows that we must have

$$a_1 = 0.$$

This equation contradicts our assumptions, and hence such a number ρ cannot serve as an index μ_1 for the series (1.92). This case is

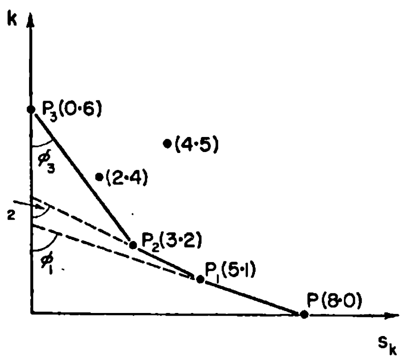

FIG. 7

represented on Fig. 6. For the determination of suitable values of ρ we proceed as follows.

Let us take a straight line and let us make it first to coincide with the horizontal axis (Fig. 7). Then commence to rotate it clockwise about the point $P_0(a_0, s_0, 0)$ (on Fig. 7 the point (8, 0)) until this line passes through some other point P_1 of our net. Then the

tangent of the acute angle, which the straight line P_0P_1 makes with the ordinate axis, determines the first possible value $\mu_1{}^{(1)}$. The points (s_k, k) of our net, through which the straight line P_0P_1 passes (there may be several of these points), determine all the terms of the series $F(z)$ which are similar to one another, and have for $\mu_1 = \mu_1{}^{(1)}$ the lowest degree. Combining these terms into a single term and equating its coefficient to zero, we obtain an equation for the determination of the quantity a_1, corresponding to the value $\mu_1 = \mu_1{}^{(1)}$.

Then we will rotate the line in the previous direction about the point P_1 (it is assumed that P_1 is the point with the greatest ordinate of the points of our net, lying on the straight line P_0P_1). We shall continue this rotation until the straight line passes through some other point of our net (it is assumed, that the point P_2 possesses the greatest ordinate of all the points of our net, lying on the straight line P_1P_2). Then the tangent of the acute angle, which the straight line P_1P_2 forms with the ordinate axis, determines the second possible value $\mu_1{}^{(2)}$. The points (s_k, k) of our net, through which the straight line P_1P_2 passes, determine all the (similar to one another) terms of the series $F(z)$, having for $\mu_1 = \mu_1{}^{(2)}$ the lowest degree. Combining these into a single term and equating its coefficients to zero, we obtain an equation for the determination of the quantity a_1, corresponding to the value $\mu_1 = \mu_1{}^{(2)}$.

Continuing this process in the way indicated, after several steps we shall obtain the straight line $P_{t-1}P_t$, passing through the point $P_t(0, p)$. With the help of this we determine the last possible value of the index $\mu_1 = \mu_1{}^{(1)}$ and then form the equation for the determination of the coefficient a_1 of the corresponding series (1.92).

The diagram which we have constructed (Fig. 7) is called *Newton's diagram or Puiseux's diagram.* We have established that:

(*a*) *The index μ_1 can have as many values as there are links in the broken line $P_0P_1 \ldots P_{t-1}P_t$.*†

(*b*) *The value $\mu_1 = \mu_1{}^{(q)}$, corresponding to the link $P_{q-1}P_q$ of this broken line, is equal to the tangent of the angle formed by the given link with the ordinate axis.*

(*c*) *In order to determine the values of the coefficients a_1 of the series* (1.92), *corresponding to the value $\mu_1 = \mu_1{}^{(q)}$, it is necessary to substitute $w = a_1 z^{\mu_1{}^{(q)}}$ in the term of the equation $\Phi(w, z) = 0$, corresponding*

† In other words, we have established, that the index μ_1 cannot have values different from those which correspond to the links of this broken line.

to the points of our net, lying on the link $P_{q-1}P_q$, to equate to zero the result of the substitution and solve the resulting equation.

Let the sum of the indicated terms of the equation have the form

$$a_{f_\sigma, g_\sigma} w^{f_\sigma} + \ldots + a_{f_1, g_1} w^{f_1} z^{g_1} \tag{1.97}$$

(we assume, that $f_\sigma > \ldots > f_1$). By virtue of our assumptions

$$\frac{g_\sigma - g_1}{f_\sigma - f_1} = \frac{g_{\sigma-1} - g_1}{f_{\sigma-1} - f_1} = \ldots = \frac{g_2 - g_1}{f_2 - f_1} = \mu_1^{(q)}. \tag{1.98}$$

Thanks to the relations (1.98) all the terms of the expression (1.97) become similar as a result of the substitution $w = a_1 z^{\mu_1}$. Hence, for the determination of the coefficient a_1 we obtain the equation

$$a_{f_\sigma, g_\sigma} a_1^{f} + \ldots + a_{f_1, g_1} a_1^{f_1} = 0, \tag{1.99}$$

or, dividing by $a_1 b_1$ (our $a_1 \neq 0$), the equation

$$a_{f_\sigma, g_\sigma} a_1^{f_\sigma - f_1} + \ldots + a_{f_1, g_1} = 0. \tag{1.100}$$

The last equation has $f_\sigma - f_1$ solutions (counting each as many times as its multiplicity). Let us note, that the quantity $f_\sigma - f_1$ is the difference between the ordinates of the points P_q and P_{q-1}. Consequently, it is equal to the projection of the link $P_{q-1}P_q$ of our broken line onto the ordinate axis.

It is immediately seen, that the sum of the projections of the links of the broken line $P_0 P_1 \ldots P_{1-1} P_t$ onto the ordinate axis is equal to p (see Fig. 7). Hence it follows that determining the coefficient a_1 from the equations (1.100), corresponding to the various $\mu_1 = \mu_1^{(1)}, \ldots \mu_1^{(t)}$, we obtain for it p values (counting each as many times as its multiplicity in the corresponding equation of (1.100)).

The calculation which we have carried out does not also permit us to conclude that all the p expressions of the form $a_1 z^{\mu_1^{(q)}}$ which have been found will be (and at that as many times as the multiplicity of a_1 in the corresponding equation (1.100)) original terms of the expansion (1.92), satisfying the equation $\Phi(w, z) = 0$. It gives us the right to assert only that our monomials can be obtained from them.

However, it turns out that the supposition just expressed is in fact true:

The number of series (1.92), *beginning with the term $a_1 z^{\mu_1^{(q)}}$ and satisfying the equation $\Phi(w, z) = 0$, is equal to the multiplicity of the quantity a_1 as a solution of the corresponding equation* (1.100).

We shall not give the proof of this theorem.

For the determination of the following terms of the series (1.92) it is necessary to make a change of variables: putting in the equation $\Phi(w, z) = 0$

$$w = \zeta^l(a_1+\nu), \qquad z = \zeta^r, \tag{1.101}$$

where $l/r = \mu_1^{(q)}$ is an irreducible fraction. As a result of this substitution we obtain for the determination of the new unknown quantity ν the equation $\Phi(\nu, z) = 0$. It is necessary to substitute in it the series

$$\nu = a_2\zeta^{\lambda_2}+{}_3a\zeta^{\lambda_3}+ \dots (\lambda_2 > \lambda_3 > \dots, a_2 \neq 0, a_3 \neq 0, \dots), \tag{1.102}$$

then we have to repeat the reasoning, which permitted us to determine the first term of the expansion (1.92). This enables us to find the expression $a_2\zeta^{\lambda_2}$. Here, if p_1 is the multiplicity of the value of a_1 taken by us (relating to the corresponding equation (1.100)), we shall obtain for a_2 precisely p_1 values (counting each one as many times as its multiplicity). These values a_2 may correspond to different values of λ_2, determined with the help of a new Newton's diagram. In order to return to the series (1.92) we, finally, make the converse substitution. Thus we find that $\mu_2 = \mu_1+\lambda_2/r$.

Continuing in this way, we can determine any number of terms of the expansions (1.92).

Example 1. Let us consider the equation

$$\Phi(w, z) = z^3w^4+w^3-8z^7 = 0.$$

It has at the point $z = 0$ the root $w = 0$ of multiplicity $p = 3$. We have to mark on the Newton's diagram for this equation (Fig. 8) the points $(7, 0)$ (corresponding to the term $-8z^7$ of the given equation) and $(0, 3)$ (corresponding to the term w^3 of the given equation). From the diagram we find that

$$\mu_1 = 7/3.$$

For the determination of the coefficient a_1 it is necessary to put $v = a_1z^{7/3}$ in the expression w^3-8z^2 and then equate to zero the result obtained. Then

$$w^3 = 8z^7 = a_1{}^3z^7-8z^7 = (a_1{}^3-8)z^7 = 0.$$

Therefore, we obtain the equation

$$a_1^3-8 = 0,$$

whence

$$a_1^{(1)} = 2, \qquad a_1^{(2)} = 2\,e^{2\pi i/3}, \qquad a_1^{(3)} = 2\,e^{4\pi i/3}.$$

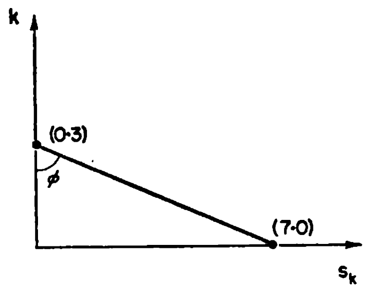

FIG. 8

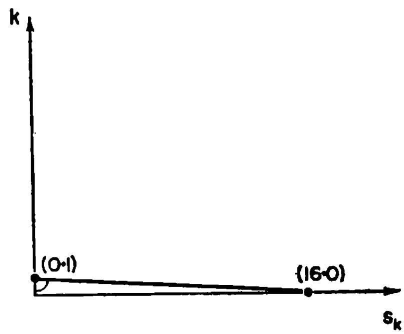

FIG. 9

For the determination of the following coefficient a_2 (we will carry out the calculation only for $a_1 = 2$) it is necessary to put

$$w = \zeta^7(2+v), \qquad z = \zeta^3.$$

Then (after dividing by ζ^{21}) the given equation is replaced by the following:

$$\Phi_1(\zeta, v) = \zeta^{16}v^4+(8\zeta^{16}+1)v^3+(24\zeta^{16}+6)v^2+$$
$$+(32\zeta^{16}+12)v+16\zeta^{16} = 0.$$

On the Newton's diagram (Fig. 9) for this equation only two points will be found: (16, 0) (corresponding to the term $16\zeta^{16}$) and (0, 1)

(corresponding to the term 12ν). Hence we find, that $\lambda_2 = 16$. In order to determine a_2 it is necessary to put $v = a_2\zeta^{16}$ in the expression $12v+16\zeta^6$ and then equate to zero the result obtained. Then we obtain the equation

$$12a_2+16 = 0,$$

whence $a_2 = -4/3$. Thus we obtain

$$w_1(z) = 2z_{PR}{}^{7/3}-4/3z_{PR}{}^{23/3}+\ldots$$

Similarly, we find that (we have put $\omega = e^{2\pi i/3}$),

$$w_2(z) = 2\omega^7 z_{PR}{}^{7/3}-\frac{4}{3}\omega^{23}z_{PR}{}^{23/3}+\ldots,$$

$$w_3(z) = 2\omega^{14} z_{PR}{}^{7/3}-\frac{4}{3}\omega^{46}z_{PR}{}^{23/3}+\ldots$$

The last three expansions can be replaced by one

$$w_I(z) = 2z^{7/3}-\frac{4}{3}z^{23/3}+\ldots$$

The point $z = 0$ turns out to be a critical point of multiplicity 3.
Remark 1. It is easy to see that

$$\left.\frac{\partial\Phi_1(\zeta, v)}{\partial v}\right|_{\substack{\zeta=0\\ v=0}} = 12 \neq 0.$$

Hence, in order to obtain the series (1.102) it is possible also to start from Theorem 3. This series must consist of integral powers and can be found by the methods indicated at the beginning of the proof of theorem 3. This is to be expected, as the equation $\Phi(w, 0) = 0$ has a root $w = 0$ of multiplicity 3, and the series for $w_1(z)$, $w_2(z)$ and $w_3(z)$ cannot contain terms of the form az^μ, where μ is an irreducible fraction with denominator greater than three.
Remark 2. The coefficients $a_2, a_3, \ldots$ of the series for the branches $w_2(z)$ and $w_3(z)$ can be determined independently of the coefficients of the series for the branch $w_1(z)$, by making use of the method just applied. However it is possible to find them without any calculations, starting from the fact that these branches comprise a single cyclical system (as the denominator μ_1 has the value 3, identical with the multiplicity of the root $w = 0$ of the equation $\Phi(w, z) = 0$).

Example 2. Let us take the equation

$$\Phi(w, z) = w^3z^2 - wz + 1 = 0,$$

considered in Art. 3 of the present chapter. There we found that the points

$$z = 0, \qquad z = 27/4, \qquad z = \infty$$

are its singular points. We will investigate the algebraic function $w(z)$ determined by the given equation in the neighbourhood of the points $z = 0$, $z = \infty$.

At the point $z = 0$ the coefficient of the highest term of the equation $\Phi(w, z) = 0$ becomes zero. Hence we shall put $w = W^{-1}$ and consider instead of the given equation

$$\chi(W, z) = W^3 - W^2z + z^2 = 0.$$

At the point $z = 0$ it has the root $W = 0$ of multiplicity 3. On the Newton's diagram of the given equation (Fig. 10) the points (0, 3), (1, 2) and (2, 0), corresponding to all its terms have to be marked. From this diagram we find that

$$\mu_1 = \tfrac{2}{3}.$$

For the determination of the coefficient a_1 it is necessary to put $W = a_1z^{2/3}$ in the expression $W^3 + z^2$ and then equate to zero the result obtained. We arrive at the equation

$$a_1^3 + 1 = 0.$$

Hence,

$$a_1^{(1)} = e^{\pi i/3}, \qquad a_1^{(2)} = e^{3\pi i/3} = -1, \qquad a^{(3)} = e^{5\pi i/3}.$$

Further calculation leads us to the analytic function

$$W_{\mathrm{I}}(z) = -z^{+2/3} + \tfrac{1}{3}z + \dots$$

In the neighbourhood of the point $z = 0$ the regular branches $W_1(z)$, $W_2(z)$ and $W_3(z)$ of this function form a cyclical system with respect to the equation $\chi(W_1 z) = 0$.

In order to return to the original algebraic function $w(z)$ we put $W = w^{-1}$. We then determine in the neighbourhood of the point $z = 0$ the algebraic function

$$w_{\mathrm{I}}(z) = -z^{-2/3}[1 + \tfrac{1}{3}z^{1/3} + \dots]^{-1} =$$
$$= -z^{-2/3}[1 + \tfrac{1}{3}z^{1/3} + \dots]$$

Thus, for the algebraic function $w(z)$, determined by the equation $\Phi(w, z) = 0$, the point $z = 0$ is a critical pole of multiplicity 3. In the neighbourhood of the point $z = 0$ the function $w(z)$ has three regular branches, which tend to infinity as $z \to 0$. The latter in their totality constitute an algebraic function $w_I(z)$ and can be defined as its regular branches.

Let us now turn to the determination of the regular branches of the algebraic function $w(z)$ in the neighbourhood of the point $z = \infty$. For this let us put $z = z_1^{-1}$, after which the given equation is replaced by the following:

$$\Psi(w, z_1) = w^3 - wz_1 + z_1^2 = 0.$$

At the point $z_1 = 0$ the last equation has the root $w = 0$ of multiplicity 3. On the Newton's diagram (Fig. 11) the points (0, 3), (1, 1)

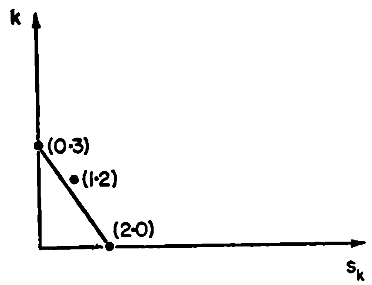

FIG. 10

and (2, 0), corresponding to all its terms have to be inserted. From this diagram we find that

$$\mu_1^{(1)} = 1,$$
$$\mu_1^{(2)} = \tfrac{1}{2}.$$

Let us find the coefficient a_1, corresponding to the value $\mu_1 = \mu_1^{(1)} = 1$. For this we have to put $w = a_1 z$ in the expression $-wz_1 + z_1^2$ and then equate the result of the substitution to zero. We obtain the equation

$$-a_1 + 1 = 0,$$

whence $a_1 = 1$.

In order to determine the following coefficient a_2 we have to put

$$w = z_1(1+v).$$

Then the equation $\Psi(w, z_1) = 0$ is replaced by the following:

$$z_1 v^3 + 3z_1 v^2 + (3z_1 - 1)v + z_1 = 0.$$

Hence, it is easy to observe that the first term of the series (1.102) is in our case equal to z_1 (that is $a_2 = 1$). Thus we obtain

$$w_{\mathrm{I}}(z_1) = w_1(z_1) = z_1 + z_1^2 + \dots$$

The function $w_1(z_1)$ is regular at the point $z_1 = 0$. It satisfies the equation $\Psi(w, z_1) = 0$ and alone constitutes a cyclical system with respect to this equation.

Now let us find the coefficient a_1, corresponding to the value $\mu_1 = \mu_1^{(2)} = \frac{1}{2}$. For this in the expression $w^3 = wz_1$ we have to

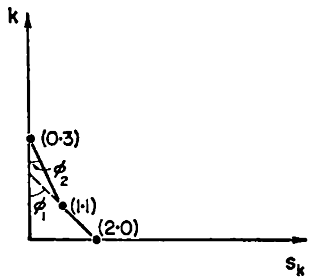

FIG. 11

put $w = az^{\frac{1}{2}}$ and equate to zero and the result obtained. We obtain the equation

$$a_1^3 - a_1 = 0,$$

or, taking into consideration that $a_1 \neq 0$, the equation

$$a_1^2 - 1 = 0,$$

hence $a_1^{(1)} = 1$, $a_1^{(2)} = -1$. In order to determine the next coefficient a_2, corresponding to the value $a_1 = a_1^{(1)} = 1$, it is necessary to put

$$w = \zeta(1+v), \qquad z_1 = \zeta^2.$$

Then instead of $\Psi(w, z_1) = 0$ we obtain the equation

$$v^3 + 3v^2 + 2v + \zeta = 0.$$

Hence it is easy to see, that the first term of the series (1.102) is

equal to $-\frac{1}{2}\zeta$ in the case considered. Thus, in the neighbourhood of the point $\zeta_1 = 0$ there is determined the analytic function

$$W_{\mathrm{II}}(z) = z_1^{1/2} - \tfrac{1}{2}z_1 + \ldots$$

In the neighbourhood of the point $z_1 = 0$ the regular branches $w_2(z_1)$ and $w_3(z_1)$ of this function also form a single cyclical system with respect to the equation $\Psi(w, z_1) = 0$.

Let us now return to the original algebraic function $w(z)$. For this it is necessary to make the inverse substitution by putting $z_1 = z^{-1}$. Then we establish, that the algebraic function $w(z)$ has in the neighbourhood of the point $z = \infty$ two cyclical systems of regular branches. The first consists of the single function

$$W_{\mathrm{I}}(z) = w_1(z) = z^{-1} + z^{-2} + \ldots,$$

the second is composed of the two regular branches of the algebraic function

$$W_{\mathrm{II}}(z) = z^{-1/2} - \tfrac{1}{2}z^{-1} + \ldots$$

Thus, at the multiple point $z = \infty$ of the given algebraic function there are located one point of regularity and one critical point of the second order.

Example 3. Let us now consider the more complicated equation†

$$\Phi(w, z) = z^3w^8 + 5z^6w^7 + (z^2-1)w^6 + 7z^5w^5 -$$
$$-4zw^4 + (-z^4 + z^3)w^3 + (z^3 + 4z^2)w^2 - z^3w + (-z^8 + 2z^7) = 0.$$

At the point $z = 0$ it has the root $w = 0$ of multiplicity $p = 6$. Let us construct its Newton's diagram (Fig. 12). We have to insert in it the following points:

(0, 6) corresponds to the term w^6
(5, 5) corresponds to the term $7z^5w^5$
(1, 4) corresponds to the term $-4zw^4$
(3, 3) corresponds to the term z^3w^3
(2, 2) corresponds to the term $4z^2w^2$
(3, 1) corresponds to the term $-z^3w$
(7, 0) corresponds to the term $2z^7$.

† As is pointed out in the book by N. Chebotarev, *Theory of Algebraic Functions* (Gostekhizdat, 1948), page 237, this equation was first considered by Young.

From this diagram we find that

$$\mu_1^{(1)} = 4, \qquad \mu_1^{(2)} = 1, \qquad \mu_1^{(3)} = \tfrac{1}{2}.$$

In order to determine the coefficient a_1, corresponding to $\mu_1 = \mu_1^{(1)} = 4$, let us equate to zero the result of the substitution of $w = a_1 z^4$, in the expression $-z^3w+2z^7$. In this case we find that $a_1 = 2$.

In order to find the coefficient a_1, corresponding to $\mu_1 = \mu_1^{(2)} = 1$, we have to put $w = a_1 z$ in the expression $4z^2w^2-z^3w$. Equating to zero the result of this substitution, we obtain that $a_1 = \frac{1}{4}$.

Finally, in order to determine the coefficient a_1 for $\mu_1 = \mu_1^{(3)} = \frac{1}{2}$ we substitute $w = a_1 z^{1/2}$ in the polynomial $w^6-4zw^4+4z^2w^2$. We must obtain zero, and hence

$$a_1^4+4a_1^2+4 = 0.$$

Hence $a_1 = \sqrt{2}$, $a_1 = -\sqrt{2}$ where each root is of multiplicity 2. Thus, two of the expansions (1.92) will begin with the term $\sqrt{(2)}z^{1/2}$

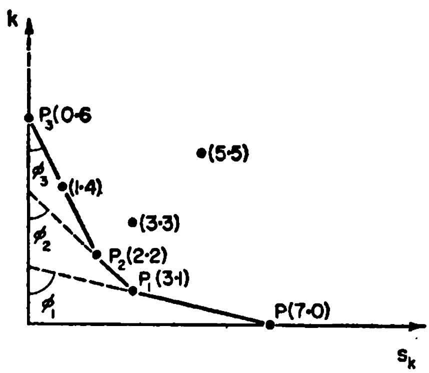

FIG. 12

and two with the term $-\sqrt{(2)}z^{1/2}$. Let us find the second terms of these expansions. For this let us first put into the given equation

$$w = \zeta(\sqrt[4]{2}+v), \qquad z = \zeta^2,$$

and then

$$w = \zeta(-\sqrt{2}+v), \qquad z = \zeta^2.$$

The corresponding calculations (which we omit) give in the first case $(\mu_1 = \frac{1}{2}, a_1 = \sqrt{2})$:

$$\lambda_2 = \tfrac{1}{4}, \qquad a_2 = \pm\sqrt[4]{(2)}/4,$$

and in the second case $(\mu_1 = \frac{1}{2}, a_1 = -\sqrt{(2)})$:

$$\lambda_2 = \tfrac{1}{4}, \qquad a_2 = \pm i\sqrt[4]{(2)}/4,$$

thus, we obtain the following series for the six regular branches of the algebraic function $w(z)$, which tend to zero as $z \to 0$:

$$W_{\mathrm{I}}(z) = w_1(z) = 2z^4 + \dots,$$

$$W_{\mathrm{II}}(z) = w_2(z) = \tfrac{1}{4}z + \dots$$

$$w_3(z) = \sqrt{(2)}z_{\mathrm{PR}}{}^{2/4} + [\sqrt[4]{(2)}/4]z_{\mathrm{PR}}{}^{3/4} + \dots$$

$$w_4(z) = -\sqrt{(2)}z_{\mathrm{PR}}{}^{2/4} + i[\sqrt[4]{(2)}/4]z_{\mathrm{PR}}{}^{3/4} + \dots$$

$$w_5(z) = \sqrt{(2)}z_{\mathrm{PR}}{}^{2/4} - i[\sqrt[4]{(2)}/4]z_{\mathrm{PR}}{}^{3/4} + \dots$$

$$w_6(z) = -\sqrt{(2)}z_{\mathrm{PR}}{}^{2/4} - i[\sqrt[4]{(2)}/4]z_{\mathrm{PR}}{}^{3/4} + \dots$$

The functions $w_3(z)$, $w_4(z)$, $w_5(z)$, $w_6(z)$ form a cyclical system with respect to the given equation. They are branches of the algebraic function

$$W_{\mathrm{III}}(z) = \sqrt{(2)}z^{2/4} + [\sqrt[4]{(2)}/4]z^{3/4} + \dots,$$

which has its branch point at the point $z = 0$.

Thus, the branches of the algebraic function $w(z)$, which tend to zero as $z \to 0$, constitute three cyclical systems. The first two each contain one function (which, naturally, is regular at the point $z = 0$ itself), the third consists of four functions.

Let us note also, that at $z = 0$ the coefficient of the highest term of the given equation becomes zero. This indicates that some of the regular branches of the algebraic function $w(z)$, determined by the equation $\Phi(w, z) = 0$, tend to infinity as $z \to 0$. Further analysis shows, that thanks to this the point $z = 0$ is also a critical pole, as the branches of the function $w(z)$, which tend to infinity, as $z \to 0$ are expanded into the series:

$$w_7(z) = -iz^{-3/2} + \dots,$$

$$w_8(z) = -iz^{-3/2} + \dots$$

We leave it to the reader to determine the next terms of these series.

9. The further study of algebraic functions

In concluding the present chapter, we wish to mention methods for the deeper study of algebraic functions.† Let $w(z)$ be an algebraic function, defined by the irreducible equation $\Phi(w, z) = 0$; let

† The exposition in the present article is necessarily of a descriptive character. A more precise treatment of the questions dealt with here would take us outside the range of this book.

$\beta_1, \ldots, \beta_r$ be the set of singular points of the function $w(z)$, and let $\mathscr{E}$ be the domain, obtained from the complete z-plane by excluding the points $\beta_1, \ldots, \beta_r$. We will continue the discussion begun in Art. 5 on the properties of such a function $w(z)$ in the whole of the domain $\mathscr{E}$.

Let us join together consecutively the points $\beta_1, \ldots, \beta_r$ by a line L free from self intersections. Let us denote by L_q the segment of the line L between the points β_q and β_{q+1}, and by E the domain obtained from the complete z-plane by the removal of the whole of the line L. It is obvious that the domain E is simply connected and comprises part of the domain $\mathscr{E}$.

Let us take a certain point $z_0 \in E$ and, in accordance with theorem 3, let us construct in its neighbourhood the regular branches $w_1(z), \ldots, w_n(z)$ of our algebraic function. Then let us continue these functions analytically throughout the whole of the domain E. We shall obtain there n regular (in particular, single-valued) functions, as continuation along any contour $\Lambda \subset E$ returns each of them to its original value. The latter happens because in one of the two parts, into which the contour Λ divides the complete z-plane, there is no singular point of the function $w(z)$. We will retain for the functions defined in this way in the domain E, their original designations: $w_1(z), \ldots, w_n(z)$. Let us note also, that on the boundary of the domain E, the line L, these functions will, generally speaking, lose their regularity, suffering a discontinuity (they will tend to different values as they approach some point of the line L from different sides).

By theorem 4 the continuation of the functions $w_k(z)$ along curves, lying in the domain $\mathscr{E}$, must make it possible to pass continuously from the values of one of these functions to the values of any other of them. Such a result can be attained only as a result of the continuation of the functions $w_k(z)$ along curves, intersecting the line L (such curves lie in the domain $\mathscr{E}$, but do not lie in the domain E). In fact, let the function $w_k(z)$ suffer a discontinuity at the point b, where our curve intersects the line L; then its values on this curve on one side of the point b must continuously follow on to the values of the function $w_s(z)$ (where $s \neq k$) on the same curve on the other side of the point b.

In order to explain many of the properties of the algebraic function $w(z)$ it is convenient to define it not on the z-plane, where it is many valued, but on a specially prepared "n-sheeted Riemann

surface", to every point of which always correspond one value of the function $w(z)$.

We will set ourselves the task of constructing the indicated surface, so that the function $w = w(z)$ maps it onto the w-plane (or on part of the w-plane) not only in a single-valued way but also continuously.

Let us arrange above the complete z-plane n other specimens of "sheets" of this plane with cuts along the line L (that is in other words, n specimens of the domain E). We will denote these sheets by the symbols $E_1, \ldots, E_n$ and place at the points of each sheet E_k corresponding to the points z the values of the function $w_k(z)$ with the same number k. Thus, with each point of the manifold, which consists of the n sheets $E_1, \ldots E_n$, there is associated one value of the function $w(z)$ by the formula

$$w(z) = w_k(z) \quad \text{for} \quad z \in E_k. \tag{1.103}$$

As a result the function $w(z)$ is single valued on this manifold which we have constructed. However, the latter will not map continuously the functions $w = w(z)$: at points of the sheet E_k close to one another, but located on different sides of the cut L, the function $w(z)$ can by virtue of formula (1.103) assume values distant from one another.

Let us take measures to ensure the continuity of the mapping $w = w(z)$. Let us take an arbitrary point ζ inside the segment L_q of the line L. Let L'_q and L''_q be the edges of the cut, made along the segment L_q in the construction of the domain E; let L'_{qk} and L''_{qk} be the edges of the corresponding cuts on the sheets E_k; and let ζ', ζ'', ζ'_{k_1} and ζ''_k be points with complex co-ordinates ζ on these edges.

Let us consider the value, to which the function $w_k(z)$ tends when the point $z \in E$ approaches the points ζ' and ζ''. We will denote them by

$$w_k(\zeta') = w(\zeta'_k); \quad w_k(\zeta'') = w(\zeta''_k)(k = 1, \ldots, n). \tag{1.104}$$

Thus, above the point ζ there are defined $2n$ values of the function $w(z)$. The point $\zeta \in \mathscr{E}$, and hence there exist exactly n different values of $w(\zeta)$. Also all the $w_k(\zeta')$ $(k = 1, \ldots, n)$ are different, as are also all the $w_k(\zeta'')$ $(k = 1, \ldots, n)$.†

† We shall be convinced of this, if the line L_q is slightly displaced so that the point ζ and the path, along which the point z approaches ζ, should turn out to be inside the new domain E. Then from the equation $w_s(\zeta') = w_t(\zeta')$ (where $s \neq t$) it is at once clear, that ζ is a singular point of the function $w(z)$. However, the latter would contradict our assumptions.

Consequently, for the points ζ and for any integer s (where $1 \leqslant s \leqslant n$) an integer t (where $1 \leqslant t \leqslant n$), can be found such that

$$w_s(\zeta') = w_t(\zeta''). \tag{1.105}$$

Let γ and δ be two interior points of the segment L_q. We shall show that $w_s(\gamma') = w_t(\gamma'')$ and $w_s(\delta') = w_t(\delta'')$. Let us assume that contrary to our assertion $w_s(\delta') = w_q(\delta'')$ and $g \neq t$. Let us consider the point $\tau \in L_q$, situated half way (along L_q) between the points γ and δ. Let $w_s(\tau') = w_h(\tau'')$. As $t = g$, then h cannot be simultaneously equal to both t and g. We shall obtain at least one pair of points; for definiteness let them be the points γ and τ, such that $w_s(\gamma') = w_t(\gamma'')$, $w_s(\tau') = w_h(\tau'')$ and $t \neq h$. Then we will take a point η half way between γ and τ and repeat the previous reasoning. Continuing the process of construction of such pairs of points indefinitely in the limit we shall find on the line L_q a point ξ, at which the quantities $w_{l_1}(\xi'')$ and $w_{l_2}(\xi'')$ will be identical with one another for $l_1 \neq l_2$ (both these quantities will be equal to $w_s(\xi')$). The latter, as we have seen, is impossible. Therefore, we arrive at the result, that in the relation (1.105) the value of the subscript t does not depend on the position of the point ζ on the segment L_q (but is determined only by the value of the subscript s).

At the points $\zeta \in L_q$ let

$$w_1(\zeta') = w_{\mu_1}(\zeta''). \tag{1.106}$$

Then we join, "stick", the edge L'_q of the cut L_q on the sheet E_1 to the edge $L_{q\mu_1}''$ of the cut L_q on the sheet E_{q_1}, and include the segment common to both sheets produced by this in the structure of the Riemann surface which we are constructing.

Then we shall consider the function $w_2(z)$, and if

$$w_2(\zeta') = w_{\mu_2}(\zeta''), \tag{1.107}$$

we stick the edge L'_{q_2} on the sheet E_2 to the edge $L''_{q\mu_2}$ on the sheet E_{μ_2} and so on. Sorting out all the functions $w_k(z)$, we shall connect all the edges L_{qk} with the corresponding edges $L''_{q\mu k}$, and we shall include the common segments which arise here in the attachment of each pair of sheets in the composition of our Riemann surface.

If in a particular case $\mu_k = k$, then in the attachment the edges L'_{qk} and L''_{qk} turn out to be connected on the one sheet E_k and the continuity of the latter on the segment L_q is established. Let us note also, that as a result of our attachments the manifold considered

will, generally speaking, have self intersections above the segment L_q.† For example, such self intersections are obtained, if we have to attach the edge L'_{q_1} to the edge L''_{q_3}, and the edge L'_{q_2} to the edge L''_{q_4}. However, in order to ensure the single-valuedness of the function $w(z)$ on the Riemann surface, we shall consider the segments along which the attachment takes place (in our example the edge L'_1 to the edge L''_3, on the one hand, and the edge L'_2 to the edge L''_4 on the other hand) to be distinct, and the resulting intersections are imaginary. Let us remember, that in fact this also happens in the construction of Riemann surfaces for other functions.‡

We will carry out such an attachment of the sheets E_k on all the segments $L_q(q = 1, \ldots r-1)$. As a result the manifold, originally consisting of n isolated sheets $E_1, \ldots, E_n$, becomes connected, transformed into a surface. The latter fact only expresses theorem 4 in a different form and follows from the irreducibility of the polynomial $\Phi(w, z)$.

We have obtained the *n-sheeted Riemann surface* **E** of the given algebraic function $w(z)$. The function $w(z)$ is single-valued and continuously changes with the continuous change of the position of the point z on the surface **E**.

The points of the sheets $E_1, \ldots, E_n$ with co-ordinates $\beta_1, \ldots, \beta_r$ have still not been included in the structure of the surface **E**. We shall add them to it; however, if in the neighbourhood of the point z the β_q branches $w_{E_1}(z), \ldots w_{E_\nu}(z)$ form a cyclical system (and only in this case), we shall consider the points with co-ordinates β_q on the sheets $E_{E_1}, \ldots, E_{E_\nu}$ as identical with one another. This convention naturally follows from the rules established above for the attachment of the edges of the cuts $L_1, \ldots, L_{r-1}$ on the sheets of the Riemann surface **E**. As a result to every critical point of the function $w(z)$ there will correspond one and only one point. Also the points with co-ordinate β_q, lying on the sheets E_k of the Riemann surface **E**, may remain different, even though the function $w(z)$ assumes identical values at them. It goes without saying, that this fact does not violate the single-valued character of the function $w = w(z)$ on the Riemann surface **E**. Let us note in connexion with this, that the inverse function $z = z(w)$, generally speaking, will not be single

† It is easy to see, that self intersections always arise with the exception of the case, where $u_k = k$ for all k (where $1 \leqslant k \leqslant n$).

‡ See for example, F.C.V., Chap. III, Art. 24.

valued: the function $w(z)$ may at different points of the Riemann surface **E** (of course, not only at points, having co-ordinates $\beta_1, \ldots \beta_r$) assume identical values.

The Riemann surface which we have constructed plays an important part in the integral calculus. The problem of calculating the integral

$$\int_{(L)} R(w, z)\,\mathrm{d}z, \tag{1.108}$$

where $w = w(z)$ is an algebraic function of z and $R(w, z)$ is a single-valued, continuous function of w and z (for the values of these variables considered) and is not defined, until it is known which branch of the function $w(z)$ must in fact be taken.† It is possible to give instructions about this, by arranging the path of integration (L) not on the z-plane, but on the Riemann surface **E** of the algebraic function $w(z)$. Then the sheet of the surface **E** itself, along which the curve (L) passes, determines the branch of the function used in the integral (1.108).

If $R(w, z)$ is a rational function of the variables w and z, the integral (1.108) is said to be an *Abelian* integral. The theory of Abelian integrals is very extensive and has important applications. The structural properties of the Riemann surfaces of algebraic functions are of first rate importance in it. The exposition of this theory would take us outside the range of the present book.‡

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† See an example of the calculation of such an integral: F.C.V., Chap. VII, §75.

‡ The reader can find a comparatively complete exposition of the theory of algebraic functions in, for example, the books: N. G. Chebotarev, *Theory of Algebraic Functions* (Gostekhizdat, 1948), P. Appell and E. Goursat, *Théorie des Fonctions Algébraiques et de leurs Intégrales*, Vol. I (Paris, Gautier–Villars, 1929). In the book by N. G. Chebotarev there is a bibliography.

CHAPTER II

DIFFERENTIAL EQUATIONS

In the present chapter we shall consider differential equations in the complex domain. We shall study the questions which arise in searching for functions, satisfying differential equations of the form

$$\frac{\mathrm{d}w}{\mathrm{d}z} = f(w, z), \tag{2.1}$$

or functions, which satisfy systems of differential equations of the form

$$\frac{\mathrm{d}w_j}{\mathrm{d}z} = f_j(w_1, \ldots, w_n, z) \qquad (j = 1, \ldots, n). \tag{2.2}$$

Here w (or $w_1, \ldots, w_n$) and z are complex variables, $f(w, z)$ (or $f_j(w_1, \ldots, w_n, z)$) are regular, or otherwise, analytic functions of their variables. The mathematical discipline, which studies such equations, bears the title of the *analytic theory of differential equations*.

10. Regular functions of two complex variables

We have already indicated, that it is intended to consider the equations (2.1) and the systems of equations (2.2), supposing that their right hand sides are regular functions. For this it is necessary first to stop in order to define certain properties of a regular function of several complex variables. In order to lighten the calculations, we shall in what follows, limit ourselves to the case of two variables. The passage to the general case of an arbitrary number of variables will not cause any difficulty.

Let $f(w, z)$ be a single-valued function of the complex variables $w = u+iv$, $z = x+iy$. We suppose that it is defined for

$$w \in C, \qquad z \in D, \tag{2.3}$$

where C is a domain on the w-plane, D is a domain on the z-plane.

The function $f(w, z)$ is said to be *continuous* at $w = w_0$, $z = z_0$ if for every number $\epsilon > 0$ it is possible to find a number $\delta > 0$, such that for

$$|w-w_0| < \delta, \quad |z-z_0| < \delta \tag{2.4}$$

we always have

$$|f(w, z)-f(w_0, z_0)| < \epsilon. \tag{2.5}$$

From this definition it follows that:

If the function $f(w, z)$ is continuous for $w = w_0$, $z = z_0$, then the function of the single variable w

$$f(w, z_0)$$

is continuous at the point $w = w_0$ and the function of the single variable z

$$f(w_0, z)$$

is continuous at the point $z = z_0$.

As is the case also for functions of real variables, the converse conclusion, generally speaking, is not true.

We here give without proof the following proposition.

If the function $f(w, z)$ is continuous for

$$w \in \bar{C}, \qquad z \in \bar{D}, \tag{2.6}$$

there exists a number $M \neq \infty$, such that for the indicated values of w and z

$$|f(w, z)| \leqslant M. \tag{2.7}$$

Here it is essential, that $\bar{C}$ and $\bar{D}$ are closed regions.

Let us now turn to the definition of a regular function.

Definition. the function $f(w, z)$ is said to be *regular* for $w = w_0$, $z = z_0$ if there exist numbers R_1 and R_2, such that

(*a*) the function $f(w, z)$ is continuous for $|w-w_0| < R_1$, $|z-z_0| < R_2$,

(*b*) $f(w, \zeta)$, for $|\zeta-z_0| < R_2$, is a regular function of w in the circle $|w-w_0| < R_1$, and $f(\omega, z)$, for $|\omega-w_0| < R_1$, is a regular function of z in the circle $|z-z_0| < R_2$.

Condition (*b*) requires the existence for all w and z satisfying the conditions

$$|w-w_0| < R_1, \quad |z-z_0| < R_2,$$

of the partial derivatives

$$\frac{\partial f}{\partial w} = \lim_{\Delta w \to 0} \frac{f(w+\Delta w,\ z)-f(w,\ z)}{\Delta w}, \tag{2.8}$$

$$\frac{\partial f}{\partial z} = \lim_{\Delta z \to 0} \frac{f(w,\ z+\Delta z)-f(w,\ z)}{\Delta z}.$$

From our definition it is seen, that a function, which is regular for $w = w_0$, $z = z_0$, will also be regular for values of n and z, close to w_0 and z_0 (in any case for n and z, satisfying the inequalities $|w-w_0| < R_1, |z-z_0| < R_2$).

It is also obvious, that for the regular function $f(w, z) = U(w, z) + iV(w, z)$ the Cauchy–Riemann conditions

$$\frac{\partial U}{\partial u} = \frac{\partial V}{\partial v}, \quad \frac{\partial U}{\partial v} = -\frac{\partial V}{\partial u}, \tag{2.9}$$

$$\frac{\partial U}{\partial x} = \frac{\partial V}{\partial y}, \quad \frac{\partial U}{\partial y} = -\frac{\partial V}{\partial x}.$$

are satisfied. Let us now suppose that the function $f(w, z)$ is regular for values w and z, satisfying the conditions (2.6). Let the line Γ be the boundary of the domain C, and the line Δ be the boundary of the domain D. Let us fix the variable z at a value corresponding to a point of the domain D, and consider in the domain C the function of the single variable w. Then by Cauchy's integral formula for a single variable function† at every point $w \in C$

$$f(w,\ z) = \frac{1}{2\pi i} \int_{\Gamma} \frac{f(\omega,\ z)}{\omega - w} d\omega. \tag{2.10}$$

By supposition $f(\omega, z)$ is a regular function of z in the closed region $\bar{D}$. Hence,

$$f(\omega,\ z) = \frac{1}{2\pi i} \int_{\Delta} \frac{f(\omega,\ \zeta)}{\zeta - z} d\zeta. \tag{2.11}$$

† See F.C.V. Chap. V, Art. 51.

Substituting this expression for $f(\omega, z)$ in equation (2.10), we finally obtain

$$f(w, z) = -\frac{1}{4\pi^2}\int\limits_{\Gamma} d\omega \int\limits_{\Delta} \frac{f(\omega, \zeta)}{(\omega - w)(\zeta - z)} d\zeta. \tag{2.12}$$

This is Cauchy's integral formula for regular functions of two complex variables.

From this formula as in the case of a single variable, there follows the existence and regularity of all the partial derivatives† of the function $f(w, z)$. From it is derived the equation

$$\frac{\partial^{p+q} f(w, z)}{\partial w^p \partial z^q} = -\frac{p!q!}{4\pi^2}\int\limits_{\Gamma} d\omega \int\limits_{\Delta} \frac{f(\omega, \zeta)}{(\omega - w)^{p+1}(\zeta - z)^{q+1}} d\zeta. \tag{2.13}$$

Using formula (2.13), it is easy to obtain a bound for the derivatives of the function $f(w, z)$‡. We shall suppose, that this function is regular for values w and z, satisfying the conditions (2.3). Let the point $w_0 \in C$, the point $z_0 \in D$, and the closed circles

$$|w - w_0| \leqslant r_1, \quad |z - z_0| \leqslant r_2 \tag{2.14}$$

consist of points of these domains. Let us denote by ν_1 and ν_2 the boundaries (circumferences) of these circles. Then by formula (2.13)

$$\frac{\partial^{p+q} f(w, z)}{\partial w^p \partial z^q} = -\frac{p!q!}{4\pi^2}\int\limits_{\gamma_1} d\omega \int\limits_{\gamma_2} \frac{f(\omega, \zeta)}{(\omega - w)^{p+1}(\zeta - z)^{q+1}} d\zeta. \tag{2.15}$$

The function $f(w, z)$ is regular, and consequently, also continuous for $|w - w_0| \leqslant r_1, |z - z_0| \leqslant r_2$. Hence there exists a number $M \neq \infty$, such that for the indicated values of w and z

$$|f(w, z)| \leqslant M.$$

Applying successively to both the integrals, forming the double integral (2.15), the theorem on the bound of an integral §, we find that

$$\left|\frac{\partial^{p+q} f}{\partial w^p \partial z^q}\right| \leqslant p!q! \frac{M}{r_1^p r_2^q}. \tag{2.16}$$

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† See F.C.V., Chap. V, formula 38.

‡ See F.C.V., Chap. V, formula 41. A similar discussion for functions of a single complex variable is to be found there.

§ See F.C.V., Chap. V, Art. 46.

The numbers r_1 and r_2 can be taken as close as desired to the radii of the greatest circles with centres at the points w_0 and z_0, located (respectively) in the domains C and D. However, it must be borne in mind that on changing the radii r_1 and r_2 in general the value of M is changed also.

In applications the following fact is often made use of.

Let $f(w, z)$ be a regular function for $w \in C, z \in D$; let L be a certain curve, lying together with its ends in the domain C; let N be a certain curve, lying together with its ends in the domain D. Then the integrals

$$F(z) = \int_L f(\omega, z)\mathrm{d}\omega, \qquad \Phi(w) = \int_N f(w, \zeta)\mathrm{d}\zeta \tag{2.17}$$

define regular functions in the domains D and C. Also the double integrals

$$\int_N F(\zeta)\mathrm{d}\zeta = \int_N \mathrm{d}\zeta \int_L f(\omega, \zeta)\mathrm{d}\omega,$$

$$\int_L \Phi(\omega)\mathrm{d}\omega = \int_L \mathrm{d}\omega \int_N f(\omega, \zeta)\mathrm{d}\zeta \tag{2.18}$$

are equal to one another.

We shall not stop for the proof of this proposition.

Weierstrass's theorem on uniformly convergent series plays an important part in the theory of regular functions of many variables. It runs as follows:

For $w \in C$, $z \in D$ let the series

$$\sum_{k-1}^{\infty} f_k(w, z) \tag{2.19}$$

converge to the function $F(w, z)$, and let its terms be regular functions. If this series converges uniformly for $w \in \bar{C}$, $z \in \bar{D}$, whenever C_1 and D_1 are some domains satisfying the conditions $\bar{C_1} \subset C$, $\bar{D_1} \subset D$, then

(a) the function $F(w, z)$ is regular for $w \in C$, $z \in D$,

(b) the derivatives $(\partial^{p+q}F)/(\partial w^p \partial z^q)$ can be calculated term by term by differentiation of the given series, and the resulting series will

converge uniformly for $w \in \bar{C}_1$, $z \in \bar{D}_1$, *if* C_1 *and* D_1 *are some domains satisfying the conditions* $\bar{C}_1 \subset C$, $\bar{D}_1 \subset D$.

This theorem is proved in exactly the same way as the similar proposition in the theory of functions of a single complex variable. However, in the proof, instead of Cauchy's integral formula for functions of one variable, formula (2.12) is naturally used.

It must be added here, that *if* L *and* N *are certain curves, lying together with their ends in the domains* C *and* D *respectively, then the integrals*

$$\int_L F(\omega, z)\mathrm{d}\omega, \qquad \int_N F(w, \zeta)\mathrm{d}\zeta \tag{2.20}$$

may be calculated by term-by-term integration, which leads to uniformly convergent (in the regions $\bar{C}_1$ *and* $\bar{D}_1$*) series.*

Double power series are encountered more often than other series in the theory of regular functions of several variables.

The series

$$\sum_{k,l=0}^{\infty} a_{k,l}(w-w_0)^k\,(z-z_0)^l \tag{2.21}$$

is known as a double power series.

At the basis of the theory of these series lies Abel's theorem, which adapted for the case considered runs:

If the series (2.21) *converges for* $w = W_0$, $z = Z_0$ *then it converges absolutely for all* w *and* z, *satisfying the inequalities*

$$|w-w_0| < R_1, \quad |z-z_0| < R_2, \tag{2.22}$$

and converges uniformly for all w *and* z, *satisfying the conditions*

$$|w-w_0| \leqslant r_1, \quad |z-z_0| \leqslant r_2. \tag{2.23}$$

Here $R_1 = |W_0-w_0|$, $R_2 = |Z_0-z_0|$; r_1 *and* r_2 *are any numbers satisfying the inequalities* $0 \leqslant r_1 < R_1$, $0 \leqslant r_2 < R_2$.

This proposition also is proved in exactly the same way, as the similar proposition in the theory of functions of one complex variable.

From the theorems of Abel and Weierstrass it follows, that the sum of the series (2.21) is a regular function of the variables w and z for $|w-w_0| < R_1, |z-z_0| < R_2$. In connexion with this there naturally arises the question of the possibility of representing the arbitrary function $F(w, z)$, regular for $w = w_0$, $z = z_0$, by a double power series of the form (2.21).

Let the function $F(w, z)$ be regular for the values w and z, which satisfy the conditions

$$|w-w_0| < R_1, \quad |z-z_0| < R_2. \tag{2.24}$$

We will construct the circles

$$L: \quad |w-w_0| = r_1, \qquad N: \quad |z-z_0| = r_2, \tag{2.25}$$

where $0 < r_1 < R_1$, $0 < r_2 < R_2$. Finally, let us locate the point w in the circle $|w-w_0| < \rho_1$, the point z in the circle $|z-z_0| < \rho_2$, where $0 < \rho_1 < r_1$, $0 < \rho_2 < r_2$. Then by formula (2.12)

$$F(w, z) = -\frac{1}{4\pi^2} \int\limits_L \frac{d\omega}{\omega - w} \int\limits_N \frac{F(\omega, \zeta)}{\zeta - z} d\zeta. \tag{2.26}$$

Subject to our conditions (as $|w-w_0/\omega-w_0| = |w-w_0|/r_1 < \rho_1/r_1 < 1$, $|z-z_0/\zeta-z_0| = |z-z_0|/r_2 < \rho_2/r_2 < 1$)

$$\begin{aligned} \frac{1}{\omega-w} &= \frac{1}{(\omega-w_0)-(w-w_0)} = \frac{1}{(\omega-w_0)\left(1-\dfrac{w-w_0}{\omega-w_0}\right)} \\ &= \frac{1}{\omega-w_0}\left[1+\frac{w-w_0}{\omega-w_0}+\left(\frac{w-w_0}{\omega-w_0}\right)^2+\dots\right], \\ \frac{1}{\zeta-z} &= \frac{1}{(\zeta-z_0)-(z-z_0)} = \frac{1}{(\zeta-z_0)\left(1-\dfrac{z-z_0}{\zeta-z_0}\right)} \\ &= \frac{1}{\zeta-z_0}\left[1+\frac{z-z_0}{\zeta-z_0}+\left(\frac{z-z_0}{\zeta-z_0}\right)^2+\dots\right]. \end{aligned} \tag{2.27}$$

Replacing in the integral (2.26) the quantities $1/(\omega-w)$ and $1/(\zeta-z)$ by the series (2.27) and using the addition to Weierstrass's theorem formulated above, we obtain the expansion

$$F(w, z) = \sum_{k,l=0}^{\infty} a_{k,l}(w-w_0)^k(z-z_0)^l, \tag{2.28}$$

where

$$a_{k,l} = -\frac{1}{4\pi^2} \int\limits_L d\omega \int\limits_N \frac{F(\omega, \zeta)}{(\omega-w_0)^{k+1}(\zeta-z_0)^{l+1}} d\zeta. \tag{2.29}$$

Comparing formulas (2.28) and (2.29) with one another, we observe that

$$a_{k,l} = \frac{1}{k!l!}\left(\frac{\partial^{k+l}F}{\partial w^k\,\partial z^l}\right)_{\substack{w = w_0 \\ z = z_0}}. \tag{2.30}$$

The series (2.28) is called *Taylor's series for the function $F(w, z)$ in the neighbourhood of the values $w = w_0$, $z = z_0$*. As is seen from the relation (2.30), the value of the double integral (2.28) does not depend on r_1 and r_2. It is not changed, if in this integral any other circles with centres at the points w_0 and z_0 and radii less than R_1 and R_2 are taken instead of the circles L and N.

As the quantities ρ_1 and ρ_2 can be taken differing from R_1 and R_2 by a quantity as small as desired, it follows that equation (2.28) will hold in every case for all values of w and z, satisfying equations (2.24).

Finally, from formula (2.30) it is possible to obtain with the help of the bound (2.16), the inequality

$$|a_{k,l}| \leqslant \frac{M}{r_1{}^k r_2{}^l}. \tag{2.31}$$

Here the numbers r_1 and r_2 can take any values sufficiently near to R_1 and R_2. However it must be kept in mind, that when r_1 and r_2 are changed, generally speaking, the magnitude of M changes also.

Therefore we have proved:

THEOREM 1. *Every function $F(w, z)$, regular for $w = w_0$, $z = z_0$ can be represented by the series* (2.28). *If the function $F(w, z)$ is regular for $|w-w_0| < R_1$, $|z-z_0| < R_2$, then the series* (2.28) *also converges for all these values of w and z. If for $|w-w_0| \leqslant r_1$, $|z-z_0| \leqslant r_2$* (where $0 < r_1 < R_1$, $0 < r_2 < R_2$)

$$|F(w, z)| \leqslant M,$$

then for the coefficients a_{kl} of the Taylor series (2.28) *the inequality* (2.31) *is satisfied.*

Let us note the following about the numbers R_1 and R_2.

The numbers R_1 and R_2, which possess the property, that the function $F(w, z)$ is regular for

$$|w-w_0| < R_1, \quad |z-z_0| < R_2,$$

but are not regular for all values of w and z, satisfying the conditions

$$|w-w_0| < R_1+\epsilon, \quad |z-z_0| < R_2+\epsilon$$

(where ϵ is an arbitrarily small positive number), are called *conjugate radii of the regular function $F(w, z)$ or conjugate radii of convergence of the series* (2.28).

We shall now introduce the concept of a meromorphic function of two complex variables.

Definition. The function $F(w, z)$ is said to be *meromorphic* for $w = w_0, z = z_0$ if for values of w and z, satisfying the conditions

$$|w-w_0| < r_1, \quad |z-z_0| < r_2 \tag{2.32}$$

(where r_1 and r_2 are any positive numbers), it can be defined by the equation

$$F(w, z) = \frac{F_1(w, z)}{F_2(w, z)}. \tag{2.33}$$

Here $F_1(w, z)$ and $F_2(w, z)$ are certain functions, regular for the indicated values of w, z and $F_2(w, z) \neq 0$.

If $F_2(w_0, z_0) \neq 0$, then for a suitable choice of r_1 and r_2 the function $F_2(w, z)$ because of its continuity will not become zero for w and z, satisfying the conditions (2.32). In this case the function $F(w, z)$ will be not only meromorphic, but also regular for $w = w_0, z = z_0$ (in general, as the function $F_2(w, z)$ can, in particular, be identically equal to unity, it follows that functions, regular for $w = w_0, z = z_0$ are special cases of functions which are meromorphic for $w = w_0$, $z = z_0$).

In what follows, when we are speaking about a function $F(w, z)$ which is meromorphic at $w = w_0, z = z_0$ we shall usually suppose, that $F_2(w_0, z_0) = 0$. Let us note, that then the function $F(w, z)$ is not defined for all the values of w and z, satisfying the conditions (2.32).

In the case where $F_2(w_0, z_0) = 0$ there are two possibilities:

(1) $F_2(w_0, z_0) = 0$, $F_1(w_0, z_0) \neq 0$. Then as $w \to w_0, z \to z_0$ the function $F(w, z) \to \infty$. In this case it is said that the values $w = w_0$, $z = z_0$ define a *pole* of the function $F(w, z)$. If the function $F(w, z)$ has a pole for $w = w_0, z = z_0$ then the function $[F(w, z]^{-1}$ is regular for these values of the variables w and z.

(2) $F_2(w_0, z_0) = F_1(w_0, z_0) = 0$ (it is here assumed, that these equations are satisfied for all† the representations of $F(w, z)$ of the

† The point is, that the representation (2..33) is not unique: the functions $F_1(w, z)$ and $F_2(w, z)$ can be multiplied by the common factor $\Phi(w, z)$ regular for $w = w_0, z = z_0$. It may, in particular, happen that $\Phi(w_0, z_0) = 0$.

form (2.33)). It is then said, that the function $F(w, z)$ has a *point of indeterminateness* for $w = w_0, z = z_0$. In this case the limiting value of the function $F(w, z)$ as $w \to w_0, z \to z_0$, depends on the way *in which the variables w and z* approach the values w_0 and z_0; the function $[F(w, z)]^{-1}$ also has a point of indeterminateness for $w = w_0, z = z_0$.

Example 1. The function

$$F(w, z) = \frac{w+1}{w+z}$$

has a pole for $w = 0, z = 0$. Let us note, that this function has poles everywhere, where $w+z = 0$ but $w \neq -1$. Thus, the poles of this function are not isolated from one another: if $w_0+z_0 = 0$ then there exist values of the variables w and z, differing from w_0 and z_0 by as little as desired, for which $w+z = 0$. This property is characteristic of functions of several complex variables (let us remember, that the poles of a function of a single variable, meromorphic in a certain domain, were isolated from one another).

Example 2. The function

$$\Phi(w, z) = \frac{w-z}{w+z}$$

has a point of indeterminateness for $w = 0, z = 0$. Let us note first, that this function does not have other points of indeterminateness (the equations $w-z = 0, w+z = 0$ do not have other common solutions). Thus, as the example chosen shows, there is not bound to be another point of indeterminateness near to one point of indeterminateness.

The remaining values of w and z, for which $w+z = 0$ (apart from $w = z = 0$), define poles of the function $\Phi\ (w, z)$. Hence in any neighbourhood of the values $w = 0, z = 0$ there are values of the variables w and z, for which the regularity of the given function is destroyed just the same (although otherwise, than at the point of indeterminateness): this property is characteristic of functions of several complex variables.

Let us put

$$\frac{w-z}{w+z} = \alpha$$

(here α is an arbitrary complex number). It is obvious, that the values $w = 0$, $z = 0$ satisfy the equation

$$w(1-\alpha)+z(1+\alpha) = 0.$$

Hence it is possible for w and z to approach zero in such a way, that in the process of their variation they will always be connected by this relation (for this, it is obviously always necessary to take $z = (\alpha-1/\alpha+1)w$). Then for all the values of w and z considered, $\Phi(w, z) = \alpha$. The quantity α is the limit of the function $\Phi(w, z)$ as w and z tend to zero in the manner indicated. Thus, any complex number may be obtained as such a limit.

In concluding the present article we will point out the method of extending the definitions of regular and meromorphic functions to the case of infinite values of the independent variables w and z. For this purpose it is necessary to use the substitutions

$$w_1 = w^{-1}, \qquad z_1 = z^{-1}. \tag{2.34}$$

As a result we obtain the functions

$$F\left(\frac{1}{w_1}, z\right), \quad F\left(w, \frac{1}{z_1}\right), \quad F\left(\frac{1}{w_1}, \frac{1}{z_1}\right).$$

Then the first of these functions is considered for the values $w_1 = 0$, $z = z_0$ (instead of the function $F(w, z)$ for $w = \infty$, $z = z_0$), the second is considered for the values $w = w_0$, $z_1 = 0$ (instead of the function $F(w, z)$ for $w = w_0$, $z = \infty$), the third is considered for the values $w_1 = 0$, $z_1 = 0$ (instead of the function $F(w, z)$ for $w = \infty$, $z = \infty$). We will not stop for further details.†

11. The solution of a differential equation with right-hand side regular for the initial values

We will once more consider the differential equation (2.1) and shall search for a function $w = w(z)$ satisfying this equation (such functions are usually called *integrals of the differential equation*) and, also, the *initial condition*

$$w(z_0) = w_0, \tag{2.35}$$

† A fuller account of the simple properties of functions of many complex variables can be found in the books: B. A. Fuchs, *Theory of the Analytic Functions of Many Complex Variables*, Chapter 1 (Gostekhizdat, 1948); S. Bochner and W. Martin, *Functions of Many Complex Variables*, Chapter II, (Gostekhizdat, 1951).

where w_0 and z_0 are certain complex numbers. We shall also call these numbers w_0 and z_0 the initial values of the variables w and z.

In the present article the right hand side of the equation (2.1)—the function $f(w, z)$—will be assumed to be regular for $w = w_0, z = z_0$.

Our immediate aim is the proof of the following proposition:

THEOREM 2. *Let $f(w, z)$ be a regular function of the variables w and z for $w = w_0$, $z = z_0$. Then there exists one and only one function $w = w(z)$, regular at the point $z = z_0$, which satisfies the equation* (2.1) *and the initial condition* (2.35).

In other words, we assert that in a certain neighbourhood of the point z_0 there exists one and only one function $w = w(z)$, representable there in the form of the power series

$$w = w_0 + w'(z_0)(z-z_0) + \frac{w''(z_0)}{2!}(z-z_0)^2 + \dots,$$

and satisfying the equation (2.1).

Proof.† In order to simplify the calculations which follow let us take the quantities

$$w - w_0, \qquad z - z_0$$

as new variables. Then the whole of the discussion is reduced to the case where $w_0 = z_0 = 0$. We will retain the previous notation for the new variables. Therefore, we shall suppose, that the function $f(w, z)$ is regular for $w = 0$, $z = 0$. We shall prove, that there exists one and only one function $w(z)$, regular at the point $z = 0$ and satisfying the conditions

$$w(0) = 0, \qquad \frac{dw}{dz} = f(w, z). \tag{2.36}$$

For values of w and z, near to zero, the function $f(w, z)$ can by theorem 1 be represented by the Taylor series (2.28). Then the equation (2.36) takes the form

$$\frac{dw}{dz} = a_{0,0} + a_{1,0}w + a_{0,1}z + a_{2,0}w^2 + a_{1,1}wz + a_{0,2}z^2 + \dots \tag{2.37}$$

† Let us note, that the method of proof of the present theorem is similar to the method used above for the proof of the existence of regular branches of an algebraic function (theorem 3 of the preceding chapter).

Let us now suppose, that we have found the required function $w = w(z)$ in the form of the series

$$w = c_1 z + c_2 z^2 + \ldots \tag{2.38}$$

(we take into account the first condition of (2.36) and take the absolute term of the series (2.38) to be equal to zero). By the second condition of (2.36) we must as a result of the substitution of the series (2.38) in the equation (2.37) obtain the identity

$$\begin{aligned} c_1 + 2c_2 z + 2c_3 z^2 + \ldots &= \\ = a_{0,0} + a_{1,0}(c_1 z + c_2 z^2 + \ldots) &+ a_{0,1} z + a_{2,0}(c_1 z + c_2 z^2 + \ldots)^2 + \\ &+ a_{1,1}(c_1 z + c_2 z^2 + \ldots) z + a_{0,2} z^2 + \ldots \end{aligned} \tag{2.39}$$

Let us note, that in order to obtain the identity (2.39) we have substituted in the left hand side of the equation (2.37) the arbitrary series (2.38).

Thanks to our assumptions, equation (2.39) is satisfied for all values of z, lying in a certain neighbourhood of the co-ordinate origin. Hence it follows, that the coefficients of identical powers of z of both sides on this identity are equal to one another. Thus, we arrive at the relations†

$$\begin{gathered} c_1 = a_{0,0}, \qquad 2c_2 = a_{1,0}c_1 + a_{0,1}, \\ 3c_3 = a_{1,0}c_2 + a_{2\,0}c_1^2 + a_{1,1}c_1 + a_{0,2}, \\ \cdots\cdots\cdots\cdots\cdots \end{gathered} \tag{2.40}$$

Hence we find successively, that

$$\begin{gathered} c_1 = a_{0,0}, \qquad c_2 = \tfrac{1}{2}(a_{1,0}a_{0,0} + a_{0,1}), \\ c_3 = \tfrac{1}{6}a_{1,0}(a_{1,0}a_{0,0} + a_{0,1}) \quad \tfrac{1}{3}a_{2,0}a_{0,0}^2 + \tfrac{1}{3}a_{1,1}a_{0,0} + \tfrac{1}{3}a_{0,2}, \\ \cdots\cdots\cdots\cdots\cdots \end{gathered} \tag{2.41}$$

Thus, if the series (2.38) satisfies the requirements of the theorem which is to be proved, then its coefficients have the values indicated by the equations (2.41). The latter uniquely define the coefficients c_k; hence there can exist only one function $w(z)$, satisfying the conditions of theorem 2.

† The power series (2.38) converges absolutely in its circle of convergence. The double power series (2.28) of the function $F(w, z)$ also converges absolutely for values of w and z, near to zero (this fact is established as a result of Abel's theorem). Hence it follows, that the operations, performed in order to obtain the relations (2.40), are justified.

However, the preceding discussion does not prove the existence of such a function: this fact was one of its initial assumptions. The proof of the indicated assertion still remains to be given.

Let us consider the series

$$w = a_{0,0}z + \tfrac{1}{2}(a_{0,0}a_{1,0} + a_{0,1})z^2 + \dots \qquad (2.42)$$

Its coefficients are the quantities, determined by the formulas (2.41). If the series (2.42) is convergent in a certain circle $|z| < R$, then the function represented by it will be the one sought: in fact, it will be regular as it is the sum of a power series. There is no doubt about the satisfaction of the first of the conditions (2.36). The satisfaction of the second condition of (2.36) follows from the fact that for the series (2.42) the relations (2.40) become identities.

In order to prove the convergence of the series (2.42) we shall apply the method of *comparison functions*. For values of the variables w and z, satisfying the conditions

$$|w| \leqslant r_1, \quad |z| \leqslant r_2$$

(we now have $w_0 = z_0 = 0$), let the function $f(w, z)$ be regular, and its modulus not exceed the positive number M. Then by theorem 1 of the present chapter

$$|a_{k,l}| \leqslant \alpha_{k,l}, \qquad (2.43)$$

where $\alpha_{k,l} = M/r_1{}^k r_2{}^l$. Here the $a_{k,l}$ are as before the coefficients of the Taylor series of the function $f(w, z)$ in the neighbourhood of the values $w = 0$, $z = 0$.

Later on, we shall consider the auxiliary differential equation

$$\frac{\mathrm{d}W}{\mathrm{d}z} = \phi(W, z) = \frac{M}{\left(1-\dfrac{W}{r_1}\right)\left(1-\dfrac{z}{r_2}\right)} =$$

$$= \alpha_{0,0} + \alpha_{1,0}W + \alpha_{0,1}z + \alpha_{2,0}W^2 + \alpha_{1,1}Wz + \alpha_{0,2}z^2 + \dots \qquad (2.44)$$

The last equation holds for $|w| < r_1$, $|z| < r_2$. Thanks to equation (2.43) $\phi(W, z)$ is a comparison function for the given function $f(w, z)$ (see the remark to the proof of theorem 3 of the preceding chapter).

We shall search for a function $W(z)$ regular at the point $z = 0$ which satisfies equation (2.44) and the condition $W(0) = 0$.

The differential equation (2.44) is easily integrated by the method of separation of the variables. Its integrals are determined from

the relations

$$W - \frac{W^2}{2r_1} = -Mr_2\ln\left(1 - \frac{z}{r_2}\right) + C. \tag{2.45}$$

Here ln is the principal branch of the logarithmic function: the consideration of its other branches is superfluous thanks to the presence in the expression (2.43) of the second component, the arbitrary constant C.

The solution of the differential equation (2.44), which becomes zero for $z = 0$, is obviously, obtained from (2.45) for $C = 0$. It has the form

$$W = W(z) = r_1 - r_1\sqrt{\left(1 + 2\frac{r_2M}{r_1}\ln\left(1 - \frac{z}{r_2}\right)\right)}. \tag{2.46}$$

Here the branch of the square root is taken, the argument of which is greater than (or equal to) 0 and less than π.

We have found a function, regular at the point $z = 0$, which satisfies equation (2.44) and the condition $W(0) = 0$. In the first part of the proof we have established, that no other such function exists.

Let us expand the function (2.46) into a series of powers of z. Let

$$W(z) = \gamma_1 z + \gamma_2 z^2 + \dots \tag{2.47}$$

The series (2.47) will converge inside the circle with centre at the point $z = 0$, which passes through the nearest (to the point $z = 0$) singular point of the function (2.46). The latter may be either the point

$$z = r_2[1 - e^{-(r_1/2r_2M)}], \tag{2.48}$$

where the expression under the square root sign in formula (2.46) becomes zero (it has a branch point there), or the point $z = r_2$, where the expression in the logarithm in the same formula becomes zero. It is obvious, that the first of these points is nearer to the co-ordinate origin. Therefore, the series (2.47) converges for

$$|z| < r_2[1 - e^{-(r_1/2r_2M)}]. \tag{2.49}$$

The series (2.47) gives the solution of the equation (2.44); hence,

repeating the discussion, which led us to equations (2.41), we obtain,

$$\gamma_1 = \alpha_{0,0}, \qquad \gamma_2 = \tfrac{1}{2}(\alpha_{1,0}\alpha_{0,0} + \alpha_{0,1}), \tag{2.50}$$

. . . .

Hence using the inequalities (2.43) and the fact that the coefficients of the series (2.42) and (2.47) are formed from the coefficients of the series (2.28) and (2.44), in accordance with formulas (2.41) and (2.50) by addition and multiplication, we find that

$$|a_{0,0}| \leqslant \gamma_1, \qquad |\tfrac{1}{2}(a_{1,0}a_{0,0} + a_{0,1}| \leqslant \gamma_2, \quad . \tag{2.51}$$

. . . .

Thus, the moduli of the coefficients of the power series (2.42) are found to be less than the coefficients of the power series (2.47), which converges in the circle (2.49). Hence by the principle of the comparision of series it follows, that the series (2.42) also converges in the circle (2.49). This proves our theorem.

Remark 1. In the course of the proof of theorem 2 it was established that the solution of equation (2.1)—the function $w = w(z)$—is represented in the neighbourhood of the point $z = z_0$ by a power series, which always converges in the circle

$$|z - z_0| < r_2[1 - e^{-(r_1/2r_2M)}]. \tag{2.52}$$

It must, however, be noted that this circle is not in any way bound to be the circle of convergence of the given power series. The latter may converge also outside its limits. The actual radius of convergence of the power series for the function $w(z)$ depends on the character of the function $f(w, z)$ and the given initial values. For example, let equation (2.1) be linear with respect to w, that is have the form

$$\frac{dw}{dz} = f_1(z) + wf_2(z), \tag{2.53}$$

where $f_1(z)$ and $f_2(z)$ are functions of a complex variable, regular in the circle $|z - z_0| < r$. We shall once more take the quantities $w - w_0$, $z - z_0$ as new variables and hence reduce our discussion to the case $w_0 = z_0 = 0$. As before, we shall retain the previous notation for the new variables.

By our assumptions the functions $f_1(z)$ and $f_2(z)$ will be represented in the circle $|z| < \rho$ by the series

$$f_1(z) = \sum_{\nu=0}^{\infty} a_\nu z^\nu, \quad f_2(z) = \sum_{\nu=0}^{\infty} b_\nu z^\nu.$$

Let $0 < r_1 < r$ and in the closed circle $|z| \leqslant r_1$ let the bounds

$$|f_1(z)| \leqslant m, \quad |f_2(z)| \leqslant m,$$

be satisfied, where m is a suitably chosen number. Then the following inequalities will hold†

$$|a_\nu| \leqslant \frac{m}{r_1{}^\nu}, \quad |b_\nu| \leqslant \frac{m}{r_1{}^\nu}. \tag{2.54}$$

Now let us consider the differential equation

$$\frac{dW}{dz} = \phi(W, z) = \frac{m(1+W)}{\left(1-\dfrac{z}{r_1}\right)} = m(1+W)\left(1+\frac{z}{r_1}+\frac{z^2}{r_1{}^2}+\dots\right). \tag{2.55}$$

The last equation holds for $|z| < r_1$ and any W. Thanks to the inequalities (2.54) $\phi(W, z)$ is a comparison function for the expression $f_1(z)+Wf_2(z)$. The equation (2.55) has the unique integral

$$W = \left(1-\frac{z}{r_1}\right)^{-mr_1} - 1,$$

which satisfies the condition $W(0) = 0$. This integral is regular at the point $z = 0$ and is represented in the circle $|z| < r_1$ by a power series. Hence it is possible (reasoning exactly the same as in the proof just derived) to conclude, that equation (2.53) has a solution (by theorem 2, unique) which is regular at the point $z = 0$, satisfying the condition $w(0) = 0$, and can also be expanded in a power series in the circle $|z| < r_1$. As the number r_1 can be taken as close as desired to r, the last series (which does not change as r_1 approaches r) will converge throughout the whole of the circle $|z| < r$.

We obtain the following proposition (in formulating it we return to the original variables; our initial values are once more the numbers w_0 and z_0).

† See F.C.V., Chap. V, Art. 46.

Let $f_1(z)$ and $f_2(z)$ be regular functions in the circle $|z-z_0| < r$. Then there exists one and only one function $w = w(z)$ regular at the point $z = z_0$, which will be regular in the circle $|z-z_0| < r$ and will satisfy equation (2.53) *and the initial condition* (2.35).

It is obvious, that in the case considered r must be taken as the distance from the point $z = z_0$ to the nearest singular point of one of the functions $f_1(z)$ and $f_2(z)$.

Let us note, that the last result does not follow directly from formula (2.52). There as $r_1 \to \infty$ the values of M and r_2 may change. Hence without further consideration it cannot be asserted, that for equation (2.53) the circle (2.52) reduces to the circle $|z-z_0| < r_2$.

Remark 2. Theorem 2 establishes, that there exists only one function $w(z)$, regular at the point $z = z_0$, which satisfies the given differential equation and the condition $w(z_0) = w_0$. However, this theorem leaves open the question of the existence of other integrals, not regular at the point z_0, but becoming w_0 for $z = z_0$ or tending to w_0, when the point z tends to the point z_2 in a specified manner. Retaining the point of view of the analytic theory of differential equations, that is assuming that w and z are complex, it is natural to limit ourselves to the consideration of analytic functions $w(z)$.† Corresponding investigations show, that the solution, indicated by theorem 2, is unique not only in the class of regular, but also in the wider class of functions which are merely analytic in the neighbourhood of the point z_0.‡ The reader can acquaint himself with the results, concerning the problem of uniqueness, in the book V. V. Golubev *Lessons on the Analytic Theory of Differential Equations* (Gostekhizdat, 1950 second edition, pages 23–29).

We shall now formulate the generalization of theorem 2 to the case of a system of differential equations (2.2).

THEOREM 3. *Let*

$$f_j(w_1, \ldots, w_n, z) \qquad (j = 1, \ldots, n)$$

be regular functions of the variables $w_1, \ldots, w_n$ for $w_1 = w_1^0, \ldots, w_n = w_n^0$ $z = z_0$. Then there exists one and only one system of functions

$$w_j = w_j(z) \qquad (j = 1, \ldots, n),$$

† For non-analytic functions $w(z)$ the differential equation $dw/dz = f(w, z)$ loses its meaning, as the derivative dw/dz does not then exist.

‡ To the latter, as we saw in Chapter I, for example, belong the algebraic functions, which can be expanded in the neighbourhood of the point $z = z_0$ into series of fractional powers of $z-z_0$.

regular at the point $z = z_0$, which satisfy the system of differential equations

$$\frac{dw_j}{dz} = f_j(w_1, \ldots, w_n, z) \qquad (j = 1, \ldots, n)$$

and the conditions

$$w_j(z_0) = w_j^{(0)} \qquad (j = 1, \ldots, n).$$

We shall omit the proof of this theorem which differs from the proof of the preceding theorem only in the complication of the individual details.

The last theorem can also be strengthened for systems of linear equations. In this case the following proposition holds.

Let

$$f_{jk}(z)(j = 1, \ldots, n;\ k = 0, 1, \ldots, n)$$

be regular functions in the circle $|z-z_0| < r$. Then there exists one and only one system of functions

$$w_j = w_j(z) \qquad (j = 1, \ldots, n),$$

regular at the point $z = z_0$ and satisfying the system of differential equations

$$\frac{dw_j}{dz} = f_{j,0}(z) + w_1 f_{j,1}(z) + \ldots + w_n f_{j,n}(z) \qquad (j = 1, \ldots, n) \tag{2.56}$$

and the initial conditions

$$w_j(z_0) = w_j^{(0)} \qquad (j = 1, \ldots, n).$$

These functions will remain regular throughout the whole of the circle $|z-z_0| < r$.

Theorem 3 can be applied to a differential equation of the n-th order

$$\frac{d^n w}{dz^n} = f\left(\frac{d^{n-1}w}{dz^{n-1}}, \ldots, \frac{dw}{dz}, w, z\right), \tag{2.57}$$

where $f(w_{n-1}, \ldots, w_1, w, z)$ is a regular function of $w_{n-1}, \ldots, w_1, w, z$ for $w_{n-1} = w_{n-1}^{(0)} \ldots w_1 = w_1^{(0)}$, $w = w_0$, $z = z_0$. Here $w_{n-1}^{(0)} \ldots$, $w_1^{(0)}$, w, z are certain complex numbers.

We will introduce the new auxiliary variables $\omega, \ldots, \omega_{n-1}$ and shall search for the system of functions regular at the point $z = z_0$

$$\omega_{n-1} = \omega_{n-1}(z), \quad \ldots, \quad \omega_1 = \omega_1(z), \quad w = w(z), \tag{2.58}$$

which satisfy the system of differential equations

$$\frac{d\omega_{n-1}}{dz} = f(\omega_{n-1}, \ldots, \omega_1, w, z), \quad \frac{d\omega_{n-2}}{dz} = \omega_{n-1}, \ldots,$$

$$\frac{d\omega_1}{dz} = \omega_2, \quad \frac{dw}{dz} = \omega_1 \tag{2.59}$$

and the conditions

$$\omega_{n-1}(z_0) = w_{n-1}^{(0)}, \quad \ldots, \quad \omega_1(z_0) = w_1^{(0)}, \quad w(z_0) = w_0. \tag{2.60}$$

By theorem 3 there exists one and only one system of functions (2.52) which satisfy the conditions enumerated. Choosing the quantities $\omega_1, \ldots, \omega_{n-1}$ and $d\omega_{n-1}/dz$ by successive differentiations of w with respect to z (this is possible from the relations (2.59)), we establish, that the function $w(z)$ found by us is the unique integral of the differential equation (2.57) regular at the point $z = z_0$, which also satisfies the initial conditions

$$w(z_0) = w_0, \quad \left.\frac{dw}{dz}\right|_{z=z_0} = w_1^{(0)}, \ldots, \left.\frac{d^{n-1}w}{dz^{n-1}}\right|_{z=z_0} = w_{n-1}^{(0)}. \tag{2.61}$$

therefore we have obtained:

THEOREM 4. *Let $f(w_{n-1}, \ldots, w_1, w, z)$ be a regular function of the variables $w_{n-1}, \ldots, w_1, w, z$ for $w_{n-1} = w_{n-1}^{(0)}, \ldots, w_1 = w_1^{(0)}$ $w = w_0$, $z = z_0$, then there exists one and only one function $w = w(z)$, regular at the point $z = z_0$, which satisfies the differential equation* (2.57) *and the initial conditions* (2.61).

Theorem 4 can also be strengthened in the case of a linear equation. The following proposition holds for it.

Let $\phi(z)$ and $f_k(z)(k = 0, 1, \ldots n-1)$ be regular functions in the circle $|z-z_0| < r$. Then there exists one and only one function $w(z)$ regular at the point $z = z_0$, which satisfies the differential equation

$$\frac{d^n w}{dz^n} = \phi(z) + f_0(z)w + f_1(z)\frac{dw}{dz} + \ldots + f_{n-1}(z)\frac{d^{n-1}w}{dz^{n-1}}$$

and the initial conditions

$$w(z_0) = w_0, \quad \left.\frac{dw}{dz}\right|_{z=z_0} = w_1^{(0)}, \ldots, \left.\frac{d^{n-1}w}{dz^{n-1}}\right|_{z=z_0} = w_{n-1}^{(0)}.$$

This function $w = w(z)$ will be regular throughout the whole of the circle $|z-z_0| < r$.

Let us now consider the problem of defining a function, which satisfies the differential equation (2.1) and one of the "infinite" initial conditions:

$$(a)\ w(\infty) = w_0, \quad (b)\ w(z_0) = \infty, \quad (c)\ w(\infty) = \infty. \qquad (2.62)$$

In each of the cases given we shall carry out a change of variables, putting respectively

$$\left.\begin{aligned} &(a)\ w = w, && z = z_1^{-1}, \\ &(b)\ w = w_1^{-1}, && z = z_0, \\ &(c)\ w = w_1^{-1}, && z = z_1^{-1}. \end{aligned}\right\} \qquad (2.63)$$

As a result we obtain instead of (2.1) the equations

$$(a)\ \frac{dw}{dz_1} = -\frac{1}{z_1^2} f\left(w, \frac{1}{z_1}\right) = F(w, z_1), \qquad (2.64)$$

$$(b)\ \frac{dw_1}{dz} = -w_1^2 f\left(\frac{1}{w_1}, w\right) = \Phi(w_1, z), \qquad (2.65)$$

$$(c)\ \frac{dw_1}{dz_1} = \frac{w_1^2}{z_1^2} f\left(\frac{1}{w_1}, \frac{1}{z_1}\right) = \Psi(w_1, z), \qquad (2.66)$$

We shall consider the problems of solving equation (2.1) with the initial conditions (a), (b) and (c) as equivalent respectively to:

(a) the problem of solving equation (2.64) with the initial condition $w = w_0$ for $z_1 = 0$;

(b) the problem of determining a function, which satisfies equation (2.65) and the initial condition $w_1(z_0) = 0$;

(c) the problem of solving equation (2.66) with the initial condition $w_1 = 0$ for $z_1 = 0$.

We shall now restrict ourselves to the consideration of these problems on the assumption, that the functions $F(w, z_1)$, $\Phi(w, z_1)$,

$\Psi(w_1, z_1)$ are regular for the corresponding initial values.† In addition to this, we shall assume, in the case of equation (2.65), that $\Phi(0, z) \not\equiv 0$, and in the case of equation (2.66), that $\Psi(0, z_1) \not\equiv 0$.

Independently of the last condition we shall obtain, using theorem 2, as the solutions of the problems formulated above the functions

$$(a)\ w(z_1) = w_0 + a_1 z_1 + a_2 z_1^2 + \dots, \tag{2.67}$$

$$(b)\ w_1(z) = (z-z_0)^l[b_0 + b_1(z-z_0) + \dots], \tag{2.68}$$

$$(c)\ w_1(z_1) = z_1^m[c_0 + c_1 z_1 + \dots]. \tag{2.69}$$

The series (2.67) and (2.69) converge in certain neighbourhoods of the point $z_1 = 0$; the series (2.68) in a certain neighbourhood of the point $z = z_0$.

Insofar as we have $\Phi(0, z) \not\equiv 0$ and $\Psi(0, z_1) \not\equiv 0$, the functions $w_1(z) \equiv 0$ and $w_1(z_1) \equiv 0$ cannot be solutions of the equations (2.65) and (2.66). Hence the function $w_1(z)$ at the point $z = z_0$, and the function $w_1(z_1)$ at the point $z_1 = 0$ have zeros of finite orders. In formulas (2.68) and (2.69) we have denoted by l and m the orders of these zeros. By our assumptions $l \geqslant 1$, $m \geqslant 1$, $b_0 \neq 0$, $c_0 \neq 0$.

Now let us return to the original variables w and z, having performed in each of the cases considered the necessary change (converse to the corresponding substitution (2.63)). As a result we obtain the solutions of equation (2.1), which correspond to the initial conditions (2.62):

$$(a)\quad w(z) = w_0 + \frac{a_1}{z} + \frac{a_2}{z^2} + \dots \tag{2.70}$$

$$\begin{aligned}(b)\ w(z) &= (z-z_0)^{-l}[b_0 + b_1(z-z_0) + \dots]^{-1} \\ &= (z-z_0)^{-l}[g_0 + g_1(z-z_0) + \dots] \\ &= \frac{g_0}{(z-z_0)^l} + \dots + \frac{g_{l-1}}{z-z_0} + g_l + g_{l+1}(z-z_0) + \dots,\end{aligned} \tag{2.71}$$

† Let us note, that these assumptions are not equivalent to the requirements of regularity of the function $f(w, z)$ for the corresponding values (a) $w = w_0$, $z = \infty$, (b) $z = z_0$, $w = \infty$, (c) $w = \infty$, $z = \infty$ as the functions F, Φ and Ψ differ from the function f by the factors indicated in formulas (2.64), (2.65) and (2.66).

$$(c)\ w(z) = \frac{1}{z_1^m}(c_0 + c_1 z_1 + \ldots)^{-1} = \frac{1}{z_1^m}(h_0 + h_1 z_1 + \ldots)$$

$$= z^m\left(h_0 + \frac{h_1}{z} + \ldots\right)$$

$$= h_0 z^m + \ldots + h_{w-1} z + h_m + \frac{h_{m+1}}{z} + \ldots \tag{2.72}$$

The series (2.70) and (2.71) converge in certain neighbourhoods of the point $z = \infty$ (circles of the form $|z| > \rho$), the series (2.71) converges in a circle of the form $|z - z_0| < \rho$. The series (2.70) defines a function, regular at the point $z = \infty$, the series (2.71) defines a function, having the point $z = z_0$ as a pole of order l, the series (2.72) defines a function, having the point $z = \infty$ as a pole of order m.

The meaning of the following assumptions is now clear:

$$\Phi(0, z) \neq 0, \qquad \Psi(0, z_1) \neq 0.$$

If it should happen, that $\Phi(0, z) \equiv 0$ (similarly in the case $\Psi(0, z_1) \equiv 0$) then the equation (2.65) would have a regular (unique by theorem 2) integral $w_1 \equiv 0$, which would satisfy the condition $w_1(z_0) = 0$. However, in this case there does not exist a function $w(z) = [w_1(z)]^{-1}$, which would be a solution of equation (2.1).

Example. Let us find a function, which satisfies the differential equation $(dw/dz) = z$ and the condition $w(i) = \infty$.

As a result of the substitution $w = w_1^{-1}$ we shall obtain instead of the given equation the following:

$$\frac{dw_1}{dz} = -w_1^2 z = \Phi(w_1, z).$$

In this case $\Phi(0, z) \equiv 0$, and the last equation has the solution $w_1 \equiv 0$, obviously, satisfying the condition $w_1(i) = 0$.

Moreover the general integral of the given equation is defined by the equation

$$w = \tfrac{1}{2}z^2 + C$$

(where C is an arbitrary constant). It is obvious, that among the functions, which comprise this general integral, there is not one which possesses a pole at the point $z = i$.

Remark. The consideration of ways of dealing with differential equations with infinite initial values is closely connected with the definition of a regular function of several variables for infinite values of its variables given above. It should be noted, that in many cases other generalizations of the idea of a regular function of several variables in the case of infinite values of its variables are found to be useful. Then it is necessary to introduce the corresponding changes in the rules for the formation of solutions of differential equations with infinite initial values given by us.

In concluding the present article let us note, that equations of the following forms have to be considered

$$F\left(\frac{dw}{dz}, w, z\right) = 0, \tag{2.73$_1$}$$

$$F\left(\frac{d^n w}{dz^n}, \ldots, \frac{dw}{dz}, w, z\right) = 0 \tag{2.73$_2$}$$

and systems of equations, not solved for the highest derivatives.

Usually an equation of the form (2.73$_1$) can be solved for (dw/dz), equations of the form (2.73$_2$) can be solved with respect to $(d^n w/dz^n)$ and hence, they can be brought into the form (2.1) and (2.57). In exactly the same way systems of equations, which are not solved with respect to the derivatives, can usually be reduced to the system considered in theorem 3 (we are speaking of systems of differential equations of the first order). After this it is possible to apply to the given equation or system of equations the theory developed in the present chapter. We have no opportunity of studying more complicated cases here.

12. The solution of a differential equation with right-hand side, having a pole for the initial values

We shall, as before, search for a function $w = w(z)$, which satisfies the differential equation (2.1) and the initial condition $w(z_0) = w_0$. However, the function $f(w, z)$ is now assumed to be meromorphic for $w = w_0$, $z = z_0$. In this case the values $w = w_0$, $z = z_0$ define a pole or point of indeterminateness of the function $f(w, z)$.

We shall now assume, that the function $f(w, z)$ has a pole for $w = w_0$, $z = z_0$. Then the function $[f(w, z)]^{-1} = \phi(w, z)$ is regular for $w = w_0$, $z = z_0$ and $\phi(w_0, z_0) = 0$. We shall exchange the parts played by the variables w and z: we shall make w the independent,

and z the dependent variable and consider the differential equation

$$\frac{dz}{dw} = [f(w, z)]^{-1} = \phi(w, z). \tag{2.74}$$

By theorem 2 there exists one and only one function $z = z(w)$, regular at the point $w = w_0$ which satisfies equation (2.74) and the initial condition $z(w_0) = z_0$. In the neighbourhood of the point $w = w_0$ this function is represented by the series

$$z(w) = z_0 + c_1(w-w_0) + c_2(w-w_0)^2 + \dots \tag{2.75}$$

From the condition $\phi(w_0, z_0) = 0$ it follows that $c_1 = 0$.† Thus, the point $w = w_0$ is a zero of the function $z(w)-z_0$ of not less than the second order.

Let us suppose in addition, that $\phi(w, z_0) \not\equiv 0$. Then the function $z(w) \equiv z_0$ cannot be an integral of equation (2.74), and the function $z(w)-z_0$ has the point $w = w_0$ as a zero of a certain finite order ν that is in the series (2.75)

$$c_1 = \dots = c_{\nu-1} = 0, \qquad c_\nu \neq 0 \qquad (\text{where } \nu \geqslant 2).$$

Subject to these conditions the series (2.75) can be put in the form

$$z - z_0 = (w-w_0)^\nu[c_\nu + c_{\nu+1}(w-w_0) + \dots]. \tag{2.76}$$

Let us take the ν-th root of both sides of equation (2.76). We obtain the equation

$$\begin{aligned}(z-z_0)^{1/\nu} &= (w-w_0)[c_\nu + c_{\nu+1}(w-w_0) + \dots]^{1/\nu}\\ &= (w-w_0)[g_1 + g_2(w-w_0) + \dots]\\ &= g_1(w-w_0) + g_2(w-w_0)^2 + \dots \end{aligned} \tag{2.77}$$

We have made use of the fact, that the function

$$[c_\nu + c_{\nu+1}(w-w_0) + \dots]^{1/\nu}$$

is regular at the point $w = w_0$ (as $c_\nu \neq 0$) and in its neighbourhood may be replaced by the series $g_1 + g_2(w-w_0) + \dots$, where $g_1 = {}^\nu c_\nu \neq 0$. As a result of the extraction of the root, in fact,

† This follows from formula (2.40) for the coefficients c_k. In order to apply the indicated formulas to the equation considered, it is necessary to take $z-z_0$ and $w-w_0$ as new variables (and $z-z_0$ must be denoted by w, and $w-w_0$ must be denoted by z) and represent equation (2.74) in the form (2.37) (that is, replace the function $\phi(w, z)$ in equation (2.74) by its Taylor series (2.28)).

we obtain on both sides of the equation (2.77) many valued functions (the various values of each of these functions are found by multiplication of some one of its values by a power of $e^{2\pi i/\nu}$). However, in order to find all the relations between w and z which arise here, it is sufficient to take into account the many-valuedness of only one of these functions. This is what we have done in writing equation (2.77): on the left of it stands a many-valued function, and on the right a single-valued function (we understand by $[c_\nu+c_{\nu+1}(w-w_0)+\ldots]^{1/\nu}$ the principal value of the corresponding many-valued function).

Our aim is to find a function $w(z)$, satisfying equation (2.77) and the condition $w(z_0) = w_0$. In equation (2.77) we will put $w-w_0 = \omega$, $(z-z_0)^{1/\nu} = \zeta$ and, using the fact that $g_1 \neq 0$, we will rewrite this equation in the form

$$\omega = g\zeta+h_2\omega^2+h_3\omega^3+\ldots = g\zeta+Q(\omega). \tag{2.78}$$

We shall search for a function $\omega = \omega(\zeta)$, regular at the point $\zeta = 0$, which satisfies equation (2.78) and the condition $\omega(0) = 0$.

Let us suppose that we have determined the required function and represented it in the form of a series

$$\omega = p_1\zeta+p_2\zeta^2+p_3\zeta^3+\ldots \tag{2.79}$$

(we take into account, that $\omega(0) = 0$ and put the absolute term of the series (2.79) equal to zero). Then, substituting the series (2.79) in place of ω in equation (2.78), we obtain an identity. Equating the coefficients of identical powers of ζ on both sides of this equation, we arrive at the relations

$$p_1 = g, \quad p_2 = h_2p_1^2, \quad p_3 = 2p_1p_2h_2+p_1^3h_3, \ldots \tag{2.80}$$

Hence

$$p_1 = g, \quad p_2 = h_2g^2, \quad p_3 = 2h_2^2g^3+h_3g^3, \ldots \tag{2.81}$$

We see, that the coefficients $p_1, p_2, p_3, \ldots$ of the series (2.79) are uniquely determined from the relations (2.80) consequently, there can exist only one function $\omega(\zeta)$, which satisfies all the conditions we have stated. The existence of a solution will be established if we prove the convergence in the neighbourhood of the point $\zeta = 0$ of the series (2.79) with the coefficients p_k, determined by formulas (2.81).

For this use can once more be made of the method of comparison functions. As the function $Q(\omega)$ of equation (2.78) is regular at

the point $\omega = 0$, there exist positive numbers r and M, such that in the closed circle $|\omega| \leqslant r$

$$|Q(\omega)| \leqslant M. \tag{2.82}$$

Then the inequalities

$$|h_k| \leqslant \frac{M}{r^k} (k = 2, 3, \ldots) \tag{2.83}$$

will hold and hence the right hand side of the equation

$$\Omega = |g|\zeta + \frac{\Omega^2}{r^2} \frac{M}{1-(\Omega/r)} \tag{2.84}$$

will be a comparison function for the function $g\zeta + Q(\omega)$.

It is easy to see, that equation (2.84) has (the unique) solution $\Omega(\zeta)$, which satisfies the condition $\Omega(0) = 0$ and can be expanded in a power series, which is convergent in the neighbourhood of the point $\zeta = 0$. Hence, proceeding exactly as in proving theorem 2, we find, that the series (2.79) (where the p_k have the values, determined from the formula (2.85)) converges in a certain neighbourhood of the point $\zeta = 0$.

Replacing in the series (2.79) ζ by $(z-z_0)^{1/\nu}$, ω by $w - w_0$, we obtain for the required integral $w(z)$ of the differential equation (2.1) the series

$$w(z) = w_0 + p_1(z-z_{01}^{1/\nu} + p_2(z-z_0)^{2/\nu} + \ldots, \tag{2.85}$$

which converges in a certain circle D with centre at the point z_0. The coefficients of this series are determined from the formulas (2.81), ν is an integer $\geqslant 2$.

The solution $w = w(z)$ which we have found is a ν-valued analytic function in the neighbourhood D of the point $z = z_0$. This function has ν regular branches, which cyclically pass into one another on continuation along any contour, lying in the domain D and passing once round the point z_0. In view of the analogy with algebraic functions we shall say, that the function $w(z)$, representable in the neighbourhood D of a certain point z_0 by a series of the form (2.85), has the latter as an *algebraic critical point of multiplicity* ν.

Therefore, we have proved the following

THEOREM 5. *Let the function $f(w, z)$ have a pole for the values $w = w_0, z = z_0$ and let $[f(w, z_0)]^{-1} \not\equiv 0$. Then in the neighbourhood of*

the point $z = z_0$ the series (2.85) (*where the coefficients p_k are determined by the formulas* (2.81), $\nu \geqslant 2$) *determines a ν-valued analytic function $w(z)$, which satisfies equation* (2.1) *and the initial condition $w(z_0) = w_0$. The point z_0 is an algebraic critical point of multiplicity ν for the function $w(z)$.*

The question of the uniqueness of the solution (2.85) of the equation (2.1) in the class of functions, analytic in the neighbourhood of the point $z = z_0$, is decided by the application of theorem 2 and the remark 2 to this theorem, to the differential equation (2.74). We shall not discuss this question.

In theorem 5 it is assumed, that $[f(w, z_0)]^{-1} = \phi(w, z_0) \not\equiv 0$. This condition is extremely material. In fact, if $\phi(w, z_0) \equiv 0$, then equation (2.74) has a regular (and unique by virtue of theorem 2) solution $z(w) \equiv z_0$, which satisfies the condition $z(w_0) = z_0$. However, the relation $z = z_0$ does not establish any kind of dependence of the variable w on the variable z. Hence for $\phi(w, z_0) \equiv 0$ the replacement of equation (2.1) by equation (2.74) will not lead us to our goal.

In this case the functions $w(z)$, which satisfy the differential equation (2.1), may have for $z = z_0$ a more complicated non-algebraic singularity.

Let us consider as an example the equation

$$\frac{dw}{dz} = \frac{1}{z}.$$

Let us search for an integral of it, which satisfies the initial condition $w(0) = 1$. It is obvious, that the function $f(w, z) = (1/z)$ has a pole for $w = 1$, $z = 0$. Interchanging the parts played by the variables, we obtain the equation

$$\frac{dz}{dw} = z,$$

which is satisfied by the function $z \equiv 0$. By theorem 2 the equation $(dz/dw) = z$ does not possess other regular solutions which satisfy the condition $z(1) = 0$. However from the equation $z = 0$ it is impossible to determine w as a function of z.

On the other hand, the general integral of the given equation is easily found and has the form

$$w = \mathrm{Ln}\, z + C.$$

These functions are infinitely many valued in the neighbourhood

of the point $z = 0$, which is their logarithmic branch point (it is sometimes called a logarithmic critical point). There is not a single one among them, which satisfies the condition $w(0) = 1$ (or even the condition $\lim_{z \to 0} w(z) = 1$). Moreover, it must not be supposed, that for $[f(w, z_0)]^{-1} \equiv 0$ the functions $w(z)$ are bound to have a non-algebraic singularity at the point $z = z_0$. In order to be convinced of the opposite, it is sufficient to consider the general integrals of the differential equations $(dw/dz) = z^{-\alpha}$, where α is equal to any real number, greater than unity.

It is important to note, that theorem 5 is applicable to the case of infinite initial values. If w_0 or z_0, or also w_0 and z_0, are infinite, then we must pass from equation (2.1) to the corresponding equation (2.63), (2.64) or (2.65) and consider it for the appropriate initial values. We can solve now the equation thus obtained not only in the case when—for the initial conditions indicated at the end of the preceding article—the function $F(w, z_1)$ or $\Phi(w_1, z)$ or $\Psi(w_1, z_1)$ is regular, but also when it has a pole. In this it is necessary to bear in mind all the conditions for the application of theorem 5.

After returning by way of the inverse substitution from equation (2.64), or (2.65), or (2.66), to equation (2.1) the function $w = w(z)$ may have at $z = z_0$ (in particular, at $z = \infty$) not only an algebraic critical point, but also an *algebraic critical pole*. In the latter case the function $w(z)$ is represented in the neighbourhood of the point $z = z_0$ (where $z_0 \neq \infty$) by a series of the form

$$w(z) = (z-z_0)^{(\lambda/\nu)}[p_0+p_1(z-z_0)^{1/\nu}+p_2(z-z_0)^{2/\nu}+\ldots], \tag{2.86}$$

or in the neighbourhood of the point $z = \infty$ by a series of the form

$$w(z) = z^{(\lambda/\nu)}[p_0+p_1z^{-1/\nu}+p_2z^{-2/\nu}+\ldots]. \tag{2.87}$$

13. Solution of a differential equation with right-hand side, having an indeterminate point for the initial values

We shall as before search for a function $w = w(z)$, which satisfies the differential equation (2.1) and the initial condition $w(z)_0 = w_0$. We shall once more suppose the function $f(w, z)$ to be meromorphic for $w = w_0$, $z = z_0$, but assume, that for these initial values it now has a point of indeterminateness.

In order to simplify the calculations we once more take $w-z_0$, $z-z_0$, as new variables, retaining for the latter the previous notations.

Therefore, in what follows, in the process of derivation (but not in formulating the final results) we always have $w_0 = z_0 = 0$.

As a first step we will consider the differential equation of the particular form

$$\frac{dw}{dz} = \frac{f(w, z)}{z},$$

where for values of w and z, near to zero, the function $f(w, z)$ is represented by the series

$$f(w, z) = \alpha w + a_{0,1}z + a_{0,2}w^2 + a_{1,1}wz + a_{0,2}z^2 + \dots \tag{2.88}$$

We will transform the given equation into the following form:

$$z\frac{dw}{dz} - \alpha w = a_{0,1}z + a_{2,0}w^2 + a_{1,1}wz + a_{0\,2}z^2 + \dots \tag{2.89}$$

It is supposed, that among the coefficients α, $a_{2,0}$, $a_{3,0}$ there are some different from zero, and, thus, the given equation belongs to the class now being studied.

Our problem is to find a function $w = w(z)$, regular at the point $z = 0$, which satisfies the differential equation (2.89) and the initial condition $w(0) = 0$. Following the usual path, let us first assume, that the required function $w(z)$ has been found in the form of the series

$$w(z) = c_1 z + c_2 z^2 + \dots \tag{2.90}$$

(considering the initial condition $w(0) = 0$, we take the absolute term of the series (2.90) to be equal to zero). Then replacing $w(z)$ in the equation (2.89) by the series (2.90), we obtain an identity. Equating to one another the coefficients of identical powers of z on both sides of it, we arrive at the relations

$$(1-\alpha)c_1 = a_{0,1}, \qquad (2-\alpha)c_2 = a_{2,0}c_1{}^2 + a_{1,1}c_1 + a_{0,2}, \dots,$$
$$(n-\alpha)c_n = q_n, \dots \tag{2.91}$$

Here q_n is a polynomial with positive (numerical) coefficients containing the quantities $a_{k,l}$ and $c_1, \dots, c_{n-1}$.

The relations (2.91), if α is not a positive integer, uniquely determine all the coefficients c_k. If however $\alpha = m$, where m is a certain positive integer, then the process of determining the coefficients c_k is broken off at the coefficient c_{m-1}. Instead of the equation for c_m we obtain the relation $q_m = 0$ between the previously

found coefficients $c_1, \ldots, c_{m-1}$. If this equation is satisfied identically (for example, for $\alpha = 1$ it turns out that $a_{0,1} = 0$ or for $\alpha = 2$ we have $a_{0,0}a_0{}^2{}_{,1} + a_{1,1}a_{0,1} + a_{0,2} = 0$ and so on), then we obtain a family of series, depending on the parameter c_m (this coefficient then also remains arbitrary), formally satisfying the differential equation (2.89) and the initial condition $w(0) = 0$. We say "formally satisfying" because these series actually define integrals of equation (2.89) only in the case of their convergence, which it is still necessary to establish. If $q_m \not\equiv 0$, the given equation does not have a regular solution which satisfies the initial condition $w(0) = 0$.

We shall suppose, that α is not a positive integer. Then it is possible to find a number A, such that for all $n = 1, 2, \ldots$

$$|n-\alpha| \geqslant A. \tag{2.92}$$

Bearing this in mind, we turn to the proof of the convergence of the series (2.90) (subject to the condition, that its coefficients are determined from the relations (2.91)).

For $|w| \leqslant r_1, |z| \leqslant r_2$ let the function $f(w, z)$ be regular, and its modulus not exceed the positive number M. Then by theorem 1 we have the bounds

$$|\alpha| \leqslant \frac{M}{r_1}, \qquad |a_{k,l}| \leqslant A_{k,l}, \tag{2.93}$$

where

$$A_{k,l} = \frac{M}{r_1{}^k r_2{}^l}.$$

Now let us consider the algebraic equation

$$AW = \frac{M}{\left(1-\dfrac{W}{r_1}\right)\left(1-\dfrac{z}{r_2}\right)} - M\left(1+\frac{W}{r_1}\right) =$$

$$= A_{0,1}z + A_{2,0}W^2 + A_{1,1}Wz + A_{0,2}z^2 + \ldots \tag{2.94}$$

(the last expansion, as usual, is correct for values of W and z, close to zero). It is easy to see, that the quadratic equation (2.94) with respect to W has one and only one solution, regular at the point $z = 0$ and satisfying the condition $W(0) = 0$. This solution is represented by the series

$$W = \gamma_1 z + \gamma_2 z^2 + \ldots \tag{2.95}$$

which is convergent in the neighbourhood of the point $z = 0$. Substituting the last expression in (2.94), we find, that the coefficients γ_k of the series (2.95) satisfy the relations

$$A\gamma_1 = A_{0,1}, \qquad A\gamma_2 = A_{2,0}\gamma_1^2 + A_{1,1}\gamma_1 + A_2, \ldots,$$
$$A\gamma_n = Q_n, \ldots,$$

where Q_n is a polynomial, constructed from the quantities $A_{k,l}$ and $\gamma_1, \ldots, \gamma_n$ in exactly the same way, as the polynomial q_n from the quantities $a_{k,l}$ and $c_1, \ldots, c_{n-1}$.

Hence, using the inequalities (2.92) and (2.93), we easily find, that for all $k = 1, 2, \ldots$

$$|c_k| \leqslant \gamma_k,$$

and consequently, the series (2.90) (with the coefficient c_k, found from the relations (2.91)) also converges in the neighbourhood of the point $z = 0$.

Therefore, we have proved:

THEOREM 6. *Let $f(w, z)$ be a regular function of the variables w and z for $w = w_0$, $z = z_0$, and the derivative $f_w'(w_0, z_0)$ not be equal to a positive integer. Then there exists one and only one function $w(z)$, regular at the point $z = z_0$, which satisfies the differential equation*

$$(z - z_0)\frac{dw}{dz} = f(w, z)$$

and the initial condition $w(z_0) = w_0$.

We shall leave it to the reader to investigate the convergence of the series (2.90) for $\alpha = m$ and $q_m \equiv 0$.

It should be noted, that theorem 6 does not decide the question of the uniqueness of the solution obtained in the class of functions, analytic in the neighbourhood of the point $z = z_0$. To obtain an exhaustive answer to this question would require the carrying out of further investigations, for which we cannot stop here.

We now set ourselves the more general task of determining the function $w = w(z)$, which satisfies the differential equation

$$\frac{dw}{dz} = \frac{f_1(w, z)}{f_2(w, z)}, \quad \text{or} \quad f_2(w, z)\frac{dw}{dz} = f_1(w, z), \qquad (2.96)$$

and the initial condition $w(0) = 0$. Here it is assumed, that the functions $f_1(w, z)$ and $f_2(w, z)$ are regular for $w = 0$, $z = 0$ and

$f_1(0, 0) = f_2(0, 0) = 0$. In addition to this, we shall assume, that for values of w and z, close to zero, these functions cannot be represented in the form

$$f_1(w, z) = w^k \bar{f}_1(w, z),$$
$$f_2(w, z) = z^l \bar{f}_2(w, z),$$

where $k, l \geqslant 0$, $k+l > 0$, and the functions $\bar{f}_1(w, z), \bar{f}_2(w, z)$ are regular for† $w = z = 0$.

Under these conditions the series

$$f_1(w, z) = a_{1,0}w + a_{0,1}z + a_{2,0}w^2 + a_{1,1}wz + a_{0,2}z^2 + \dots \qquad (2.97)$$

must contain terms, depending only on the variable z, and the series

$$f_2(w, z) = b_{1,0}w + b_{0,1}z + b_{2,0}w^2 + b_{1,1}wz^2 + b_{0,2}z^2 + \dots \qquad (2.98)$$

terms, depending only on the variable w.

Let us consider the terms of the series (2.97), which depend only on w (if there are any such). Let $a_{m,0}w^m$ be the lowest of these (in other words, it is assumed, that $a_{1,0} = \dots = a_{m-1,0} = 0$, $a_{m,0} \neq 0$). Then let us take terms of this series, containing w to the $m-1$-th degree. Let $a_{m-1,r_1}w^{m-1}z^{r_1}$ be the lowest of them. Then we, proceeding on the same principle, will select from the series considered, the term $a_{m-2,r_2}w^{m-2}z^{r_2}$ and so on. This process finishes with the selection of the term $a_{0,r_m}z^{r_m}$ (that is, it is supposed, that $a_{0,1} = \dots = a_{0,r_m-1} = 0$, $a_{0,r_m} \neq 0$). As a result we construct a system of terms of the series (2.97):

$$a_{m,0}w^m, \qquad a_{m-1,r_1}w^{m-1}z^{r_1}, \dots, a_{0,r_m}z^{r_m}. \qquad (2.99)$$

Let

$$b_{n,0}w^n, \qquad b_{n-1,s_1}w^{n-1}z^{s_1}, \dots, b_{0,s_n}z^{s_n} \qquad (2.100)$$

be a similar set of terms of the series (2.98).

It must be remembered, that in accordance with our assumptions, the series (2.97) must contain terms, which depend on z alone, the series (2.98) must contain terms, which depend on w only. Hence the presence in the sets (2.99) and (2.100) of the terms $a_{0,r_m}z^{r_m}$, $b_{n,0}w^n$ is guaranteed, however any other term, indicated in (2.99) and (2.100), may in reality be absent.

† The last condition is equivalent to the supposition that

$$f_1(0, z) \not\equiv 0, \qquad f_2(w, 0) \not\equiv 0.$$

We shall attempt to find a solution of the given problem by representing the variables w and z in the form

$$z = \zeta^\nu, \qquad w = \zeta^\lambda \omega, \tag{2.101}$$

where $\omega = \omega(\zeta)$. Here the function $\omega(\zeta)$ must be regular at the point $\zeta = 0$, and $\omega(0)$ must be different from zero; ν and λ are positive numbers, still subject to definition (it is obvious, that insofar as the choice of the number λ is at our disposal, the consideration of zero values of the quantity ω_0 has no meaning).

Then in the neighbourhood of the point $\zeta = 0$

$$\omega = \omega(\zeta) = \omega_0 + p_1\zeta + p_2\zeta^2 + \dots \tag{2.102}$$

the specification of the variables w and z by formulas (2.101) (together with the assumption on the possibility of representing the function $\omega(\zeta)$ by the series (2.102)) guarantees the satisfaction of the initial condition $w(0) = 0$. Our task is to select the quantities ν, λ, ω so that the function $w = w(z)$, determined from formula (2.101), satisfies equation (2.96). Let us substitute the expressions (2.101) for w and z in the last equation. It will then assume the form

$$f_2(\zeta^\lambda\omega, \zeta^\nu)\frac{\mathrm{d}w}{\mathrm{d}\zeta} = f_1(\zeta^\lambda\omega, \zeta^\nu),$$

or (we substitute here $(\mathrm{d}w/\mathrm{d}\zeta)$ and $(\mathrm{d}z/\mathrm{d}\zeta)$ from the equations (2.101) and multiply both sides of the resulting relation by ζ)

$$f_2(\zeta^\lambda\omega, \zeta^\nu)\left(\lambda\zeta^\lambda\omega + \zeta^{\lambda+1}\frac{\mathrm{d}\omega}{\mathrm{d}\zeta}\right) = f_1(\zeta^\lambda\omega, \zeta^\nu)\nu\zeta^\nu. \tag{2.103}$$

If the equations (2.101) together with the series (2.102) for the function $\omega(\zeta)$ determine a solution of the differential equation (2.96), then the relation (2.103) is satisfied identically: the coefficients of the same powers of ζ of both sides will be equal to one another. Calculating these equations, beginning with the least power of ζ, we obtain equations, which must be satisfied by a suitable choice of the coefficients of the series (2.102).

However so long as we do not know what λ and ν are equal to, we cannot say which terms of the series, taking part in equation (2.103) are in fact the least, that is contain ζ to the lowest degree. It can only be asserted, that they are obtained, when we take-in the series for $f_1(\zeta^\lambda\omega, \zeta^\nu)$ and $f_2(\zeta^\lambda\omega, \zeta^\nu)$ one of the terms, which

result from the substitution of (2.101) into the terms in (2.99) and (2.100); in the series (2.102) for $\omega(\zeta)$—the absolute term; and in the expression

$$\lambda\zeta^{\lambda}\omega+\zeta^{\lambda+1}\frac{d\omega}{d\zeta} \tag{2.104}$$

—the monomial $\lambda\omega_0\zeta^{\lambda}$. Hence all the terms, which are of lowest degree and are found in the relation (2.103) on the left, contain ω_0.

Among the terms of the relation (2.103), which are present on the right-hand side and can be of lowest degree, only the term

$$a_{0,r_m}(\zeta^{\nu})^{r_m}\nu\zeta^{\nu}=\nu a_{0,r_m}\zeta^{\nu(r_m+1)}$$

does not contain ω_0. The quantities ω_0 and $a_{0,\,r_m}$ are by our assumptions different from zero. Hence in the relation (2.103) there must be, at least, two lowest terms, having for the given choice of ν and λ identical powers of ζ. This fact leads us to the idea of the possibility of using Newton's diagram to determine the numbers ν and λ.

Passing to its construction, we have first to mark on this diagram the points, which correspond (according to the rules, given in Art. 8 Chapter I) to the terms of the relation (2.103), which can be of lowest degree. As the co-ordinates of these points it is possible to use the degrees of the terms of the corresponding series (before the substitution (2.101)) with respect to z (abscissa) and with respect to w (ordinate).

The terms of the series for the function $f_1(\zeta^{\lambda}\omega,\zeta^{\nu})$ are multiplied in the relation (2.103) by the expression $\nu\zeta^{\nu}$, equivalent for the determination of the degrees of the resulting terms (in virtue of the first formula of (2.101)) to the first power of z. Hence the first group of points P_j, marked on the diagram, must correspond to the monomials of the set (2.99) multiplied by νz (where for the formation of the lowest terms of the relation (2.103) it is necessary to replace in them w by $\omega_0\zeta^{\lambda}$, z by ζ^{ν}):

$$\nu a_{m,0}w^m z,\qquad \nu a_{m-1,r_1}w^{m-1}z^{r_1+1},\ \ldots,\qquad \nu a_{0,r_m}z^{r_m+1}. \tag{2.105}$$

The terms of the series for the function $f_2(\zeta^{\lambda}\omega,\zeta^{\nu})$ are multiplied in the relation (2.103) by the expression (2.104) with lowest term $\lambda\omega_0\zeta^{\lambda}$, equivalent for the determination of the degrees of the resulting terms (in virtue of the second formula of (2.101)) to the first power

of w. Hence the second group of points Q_j, marked on the diagram, must correspond to the monomials of the set (2.100), multiplied by λw (where in the formation of the terms of lowest degree of the relation (2.103) it is necessary to replace in them w by $\omega_0\zeta^\lambda$, z by ζ^ν):

$$\lambda b_{n,0}w^{n+1}, \quad \lambda b_{n-1,s_1}w^n z^{s_1}, \ldots, \lambda b_{0,s_n}wz^{s_n}. \tag{2.106}$$

Therefore, on the diagram there are marked the points

$$\begin{array}{llll} P_0(1, m), & P_1(r_1+1, m-1), & \ldots, & P_m(r_m+1, 0), \\ Q_0(0, n+1), & Q_1(s_1, n), & \ldots, & Q_n(s_n, 1). \end{array} \tag{2.107}$$

As indicated above, the systems (2.99) and (2.100) may not contain all the terms indicated in the corresponding formulas. Hence on the diagram it may turn out that in fact not all the points, indicated in the enumeration (2.107) are present.

We shall now construct (by the method indicated in Art. 8 Chapter I) the convex broken line, consisting of the straight lines "nearest" to the co-ordinate origin, which pass through the points of the set (2.107). This broken line will finish at the points P_m (lying on the axis of the abscissa) and Q_0 (lying on the ordinate axis). Then:

(1) *The tangents of the angles of inclination of the links of this broken line to the ordinate axis determine the possible values of the fraction λ/ν (this fraction is made irreducible, its numerator is taken as λ, its denominator as ν).*

(2) *The points P_j and Q_j, lying on one link, determine the terms of the relation* (2.103) *which are lowest and similar to one another in the corresponding substitution* (2.101). *Equating their sum (with appropriate signs) to zero, we obtain a relation, which must serve to determine the quantity ω_0.*

In order to avoid misunderstanding, it must be noted, that in our case the Newton diagram can have links of zero length. This will take place, if (a) some of the points $P_0, \ldots, P_{m-1}$ are identical with some of the points $Q_1, \ldots, Q_n$ and (b) the line, which we use to construct the Newton's diagram, after one of its rotations passes through such a "double" point (the latter turning out to be the vertex of a link, and not an interior point of it). Such a link of zero length does not make an angle with the ordinate axis and does not give a definite value for the quantity λ/ν. We shall leave this case aside.

Thus sets of possible values of the quantities λ, ν and ω_0 have been found.† Every such set determines a certain substitution of the variables w and z by the formulas (2.101). We shall not continue the equation of the coefficients in the relation (2.103) and shall give it the form

$$\zeta\frac{d\omega}{d\zeta} = \frac{\nu f_1(\zeta^\lambda\omega,\ \zeta^\nu)\zeta^{\nu-\lambda} - \lambda\omega f_2(\zeta^\lambda\omega,\ \zeta^\nu)}{f_2(\zeta^\lambda\omega,\ \zeta^\nu)} = F(\omega,\ z). \qquad (2.108)$$

We have obtained a differential equation for the function $\omega(\zeta)$ with the initial condition $\omega(0) = \omega_0$ (ω_0 must be different from zero). If the function $F(\omega, \zeta)$ is regular for $\omega = \omega_0$, $\zeta = 0$ and satisfies the other conditions of theorem 6, then by this theorem ω is determined in the neighbourhood of the point $\zeta = 0$ as a regular function of ζ. Then, making the change of variables, converse to the substitution (2.101), we shall obtain w in the neighbourhood of the point $z = 0$ as a ν-valued analytic function of the variable z. If for the function $F(\omega, \zeta)$ the conditions of theorem 6 (or the altered conditions of theorem 6 with $\alpha = m$, $q_m = 0$) are not satisfied, then for the values of λ, ν and ω_0 considered, the solutions obtained by our method of the equation (2.96), having the form (2.101), turn out to be impossible (or the repeated application of our method turns out to be necessary for this).

Generally speaking, the method indicated by us enables us to obtain some solutions of the differential equations (2.96), many valued and analytic in the neighbourhood of the point $z = 0$, which also satisfy the initial condition $w(0) = 0$. The functions obtained in this way will either be regular at the point z_0 or will have an algebraic critical point at it.

In the theory of differential equations it is shown that in the cases considered it is (with certain exceptions) possible to find by the method indicated, all the solutions of the differential equation which have an algebraic critical point at the point $z = 0$.‡

We shall not stop in order to pass from the initial condition $w(0) = 0$ to the more general condition $w(z_0) = w_0$. We merely mention that in this case also it is possible to consider infinite initial values of the variables w and z (by the method indicated at the end

† We exclude the set of values λ, ν, ω_0 with $\omega_0 = 0$ from further consideration, as not satisfying the stated conditions.

‡ Further details can be found in the book: E. Ince, *Ordinary Differential Equations* (Kharkhov, 1939). In this book there is a bibliography.

of Art. 11 of the present chapter). Here the point $z = z_0$, and in particular, the point $z = \infty$, may be an algebraic critical pole of the function $w(z)$.

Example 1. Let it be required to find the function, which satisfies the differential equation

$$\frac{dw}{dz} = \frac{z^2}{w^2} = f(w, z)$$

and the initial condition $w(0) = 0$. This function $f(w, z)$ has a point of indeterminateness at $w = 0$, $z = 0$; the other conditions for the applicability of the method which has just been analysed are also satisfied.

The general integral of the given equation

$$w = \sqrt{2}(\tfrac{1}{3}z^3 + C)^{1/4},$$

and with its help the required particular solution

$$w = \sqrt[4]{(4/3)}z^{3/4}$$

are found without any difficulty. However, in order to illustrate the method analysed above we shall apply the latter to the given equation. In order to make the substitution (2.101), we have first of all to find the quantities λ, ν and ω_0. The sets (2.105) and (2.106) consist in our case of the two monomials:

$$\nu z^3, \qquad \lambda w^4.$$

On the Newton's diagram (Fig. 13) two points are marked; P (3, 0) and $Q(0, 4)$. Hence $(\lambda/\nu) = \frac{3}{4}$, $\lambda = 3$, $\nu = 4$.

For the determination of the quantity ω_0 we obtain the equation

$$\lambda w^4 = \nu z^3,$$

where it is necessary to put $\nu = 4$, $\lambda = 3$, $z = \zeta^4$, $w = \omega_0\zeta^3$. We find from this, that $\omega_0 = \sqrt[4]{(4/3)}$.

In the given differential equation we now make the change of variables

$$z = \zeta^4, \qquad w = \zeta^3\omega.$$

As a result we obtain in place of the given equation

$$\zeta\frac{dw}{d\zeta} = \frac{4 - 3\omega^4}{\omega^2} = F(\omega, \zeta).$$

We have to look for an integral of it, equal to $\omega_0 = \sqrt[4]{(4/3)}$ for $\zeta = 0$. The function $F(\omega, \zeta)$ is regular for $\omega = \sqrt[4]{(4/3)}$ and $\zeta = 0$,

$$F[\sqrt[4]{(4/3)}, 0] = 0, \quad F'_{\omega}[\sqrt[4]{(4/3)}, 0] = -4\sqrt[4]{108}$$

(is not a positive integer). Hence, theorem 6 is applicable, and we can obtain a solution of the latter equation in the form of the series (2.90). However, it is immediately seen that the function $\omega \equiv \sqrt[4]{(4/3)}$ satisfies both the differential equation considered and the initial condition. By virtue of theorem 6 this equation does not possess

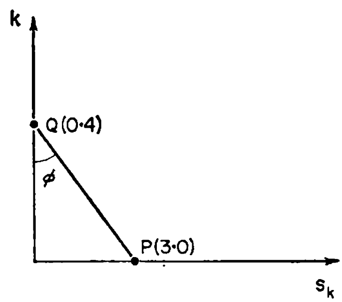

FIG. 13

other regular solutions, which satisfy the condition $\omega(0) = \sqrt[4]{(4/3)}$. As a result of the converse substitution we again obtain the previously found integral $w = \sqrt[4]{(4/3)}z^{3/4}$

Example 2. Find the function which satisfies the differential equation

$$\frac{dw}{dz} = \frac{w^3 + z^3}{w^4 + bz^2} = f(w, z)$$

and the initial condition $w(0) = 0$. The function $f(w, z)$ has a point of indeterminateness for $w = 0$, $z = 0$. The other conditions for the application of the method analysed are also satisfied.

In order to make the substitution (2.101), it is necessary first to find the quantities λ, ν and ω_0. In our case the sets (2.105) and (2.106) consist of the monomials

$$\nu w^3 z, \qquad \nu z^4,$$

$$\lambda w^5, \qquad \lambda b w z^2.$$

On the Newton's diagram (Fig. 14) four points are marked: P_0 (1,3), $P_1(4, 0)$, $Q_0(0, 5)$ and $Q_1(2, 1)$. From this the Newton's diagram

consists of two links P_1Q_1 and Q_1Q_0 (on this link there is also the point P_0). Hence we find the possible values of λ and ν:

(1) $\lambda = 2$, $\nu = 1$ (for the link P_1Q_1);
(2) $\lambda = 1$, $\nu = 2$ (for the link Q_1Q_0).

In the first case for the determination of the quantity ω_0 we obtain the equation

$$\lambda bwz^2 = \nu z^4,$$

where it is necessary to put $\lambda = 2$, $\nu = 1$, $z = \zeta$, $w = \omega_0\zeta^2$. Hence we find, that $\omega_0 = (2b)^{-1}$. In the case $b = 0$ this set of values of λ, ν, ω_0 does not occur.

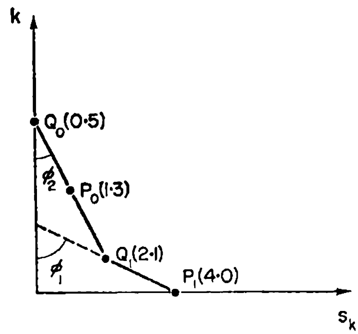

FIG. 14

In the second case for the determination of the quantity ω_0 we obtain the equation

$$\lambda w^5 + \lambda bwz^2 = \nu w^3 z,$$

where it is necessary to put $\lambda = 1$, $\nu = 2$, $w = \omega_0\zeta$, $z = \zeta^2$. After the indicated substitutions and division by ω_0 (which it is permissible for us to do, as $\omega_0 \neq 0$), this equation assumes the form

$$\omega_0^4 - 2\omega_0^2 + b = 0.$$

Hence we obtain four values for ω_0:†

$$\omega_0^{(1)} = \sqrt{[1+\sqrt{(1-b)}]}, \qquad \omega_0^{(2)} = \sqrt{[1-\sqrt{(1-b)}]},$$
$$\omega_0^{(3)} = -\sqrt{[1+\sqrt{(1-b)}]}, \qquad \omega_0^{(4)} = -\sqrt{[1-\sqrt{(1-b)}]}.$$

† In the case $b = 0$ two of them, namely $\omega_0^{(2)}$ and $\omega_0^{(4)}$ must be immediately excluded. In the last equations the square root has a value, determined by the equation

$$\sqrt{A} = \exp(\tfrac{1}{2} \ln A).$$

In the first case we carry out in the given differential equation the substitution

$$w = \zeta^2 \omega, \qquad z = \zeta.$$

As a result we obtain the equation

$$\zeta \frac{d\omega}{d\zeta} = \frac{-2\zeta^3\omega^3 + \zeta^2\omega^2 - 2b + 1}{\omega^2\zeta^3 + b} = F(\omega, \zeta).$$

We have to find an integral of it which is equal to $(2b)^{-1}$ for $\zeta = 0$. If $b \neq 0$, the function $F(\omega, \zeta)$ is regular for $\zeta = 0$ and $\omega = (2b)^{-1}$ $F(1/2b, 0) = 0$, $F_\omega'(1/2b, 0) \neq 0$.

Hence for the given initial condition this equation has, if $b \neq 0$, a unique regular solution. Therefore, we have found that for $b \neq 0$ the given equation has a solution (which also satisfies the initial condition $w(0) = 0$):

$$W_{\mathrm{I}}(z) = (2b)^{-1} z^2 + \dots$$

For the set of values $\lambda = 1$, $\nu = 2$, $\omega_0 = \omega_0^{(j)}$ $(j = 1,2,3,4)$ we carry out in the given differential equation the change of variables

$$z = \zeta^2, \qquad w = \zeta\omega.$$

As a result we obtain the equation

$$\zeta \frac{d\omega}{d\zeta} = \frac{-\omega^5 + 2\omega^3 - \omega b + 2\zeta^3}{\omega^4 + b} = \Phi(\omega, \zeta).$$

We have to find an integral of it equal to $\omega_0^{(j)}$ for $\zeta = 0$. If $b \neq 0$ the function $\Phi(w, \zeta)$ is regular for $\zeta = 0$ and $\omega = \omega_0^{(j)}$,

$$F(\omega_0^{(j)}, 0) = 0,$$

$$F_\omega'(\pm\sqrt{[1 \pm \sqrt{(1-b)}]}, 0) = -2[1 \pm \sqrt{(1-b)}].$$

Hence by theorem 6, if $b \neq 0$ or $-(m + m^2/4)$, where m is some positive integer, this equation has for each initial condition $\omega(0) = \omega_0^{(j)}$ one regular solution. Hence, for $b \neq -(m + m^2/4)$, where m is some positive integer, we find four more solutions of the given equation, satisfying the initial condition:

$$W_{\mathrm{II}}(z) = \omega_0^{(1)} z^{1/2} + \dots; \qquad W_{\mathrm{IV}}(z) = \omega_0^{(3)} z^{1/2} + \dots;$$

$$W_{\mathrm{III}}(z) = \omega_0^{(2)} z^{1/2} + \dots; \qquad W_{\mathrm{V}}(z) = \omega_0^{(4)} z^{1/2} + \dots;$$

where $W_{\mathrm{II}}(z), \dots, W_{\mathrm{V}}(z)$ are two valued analytic functions in the neighbourhood of the point $z = 0$.

In conclusion we once more emphasize that we have no basis for asserting that we have found all the functions analytic in the neighbourhood of the point $z = 0$, which satisfy there the given differential equation and the initial condition $w(0) = 0$.

Example 3. We have shown above that an equation of the form $(\mathrm{d}w/\mathrm{d}z) = f(w, z)$ where the function $f(w, z)$ which has a point of indeterminateness for $w = 0$, $z = 0$, does not belong to the class which we have considered, may nevertheless have, a solution $w = w(z)$ regular at the point $z = 0$, which satisfies the initial condition $w(0) = 0$. As an example we have here the differential equation

$$\frac{\mathrm{d}w}{\mathrm{d}z} = \frac{w}{z}.$$

Its general integral is of the form $w = Cz$, where C is an arbitrary constant number. Thus, the given equation has an infinite set of regular solutions, satisfying the initial condition $w(0) = 0$.

14. Linear differential equations of the second order

In the present article linear differential equations of the second order are considered.

$$w'' + f_1(z)w' + f_0(z)w = f(z). \tag{2.109}$$

Let us note that the results obtained here can be extended to linear equations of higher orders. We shall limit ourselves to differential equations of the second order, so as to avoid complicated and lengthy calculations.

In Art. 11 of the present chapter we established (see the supplement to theorem 4):

If the functions $f(z), f_0(z)$ and $f_1(z)$ are regular in the circle $|z-z_0| < r$, then there exists one and only one function $w = w(z)$, regular at the point $z = z_0$, which satisfies equation (2.109) and the initial condition

$$w(z_0) = w_0, \qquad w'(z_0) = w_0' \tag{2.110}$$

(where w_0 and w_0' are any complex constants). This function $w = w(z)$ is regular throughout the whole of the circle $|z-z_0| < r$.

In this proposition the circle $|z-z_0| < r$ can be taken as maximal, that is such that on the circumference $|z-z_0| = r$ there will occur singular points of at least one of the functions $f_1(z), f_0(z)$ and $f(z)$.

The function $w(z)$ may also be continued analytically outside the boundary of the circle $|z-z_0| < r$. For this it is necessary to find an integral $W(z)$ of the differential equation (2.109), regular at the point $z = z_1$ (where z_1 is some point of the circle $|z-z_0| < r$, which is not identical with the point z_0) and satisfies the initial conditions

$$W(z_1) = w(z_1) = w_1, \qquad W'(z_1) = w'(z_1) = w_1'. \tag{2.111}$$

The function $W(z)$ obtained in this way will be regular in the circle $|z-z_1| < r_1$, where the radius r_1 is again taken to be maximal. In the common part of the circles $|z-z_0| < r$ and $|z-z_1| < r_1$ there exists only one regular function, which satisfies the differential equation (2.109) and the conditions (2.111). Hence the function $W(z)$ is the analytic continuation of the function $w(z)$ into the part of the circle $|z-z_1| < r_1$, which lies outside the circle $|z-z_0| < r$.

Repeated application of the process of analytic continuation enables us to continue the function $w(z)$ from the neighbourhood of the point z_0 to the whole of the domain $\mathscr{E}$ of common regularity of the functions $f(z)$, $f_0(z)$ and $f_1(z)$ which is adjacent to this point. We will determine in this domain an analytic (generally speaking, many valued) function, which for the present we will denote by the symbol $w(z)$.

By the method of construction of the domain $\mathscr{E}$ its boundary consists of singular points of the functions $f(z)$, $f_0(z)$ and $f_1(z)$. Usually the points of the boundary of the domain $\mathscr{E}$ are singular points also of the function $w(z)$, although it is not possible to assert that all the points of this boundary are singular points of all the integrals of the differential equation (2.109). But, neglecting here the possible exceptions, it can be said, that the position of the indicated singular points (as they are points of the boundary of the domain $\mathscr{E}$) does not depend on the arbitrarily specified values of the constants w_0 and w_0', which determine the choice of the original regular integral $w(z)$ in the neighbourhood of the point $z = z_0$. On changing w_0 and w_0', we shall, generally speaking, obtain in the domain $\mathscr{E}$ different analytic functions, satisfying the differential equation (2.109).† As a rule they will have the same singular

† Moreover, it may happen, that on giving new values to the constants w_0 and w_0', we obtain in the neighbourhood of the point $z = z_0$ a regular integral of equation (2.109), which is also obtained by means of analytic continuation from the integral $w(z)$ and thus belongs to the previous analytic function.

points. In those cases, where the set of singular points of the integral of the differential equation is not changed on the replacement of some of its initial conditions by others (at a fixed point), these singular points are said to be *fixed*. We have arrived at the result, *that the singular points of the integrals of linear differential equations of the second order are fixed.*

The state of affairs is quite different for non-linear differential equations. For example, the integral of the differential equation $w' = w^2$, which at the point $z = z_0$ assumes the value w_0, is defined by the equation

$$w = \frac{w_0}{1-w_0(z-z_0)}.$$

This function (for $w_0 \neq 0$) has a pole of the first order at the point $z = z_0+w_0^{-1}$. It is obvious, that the position of this pole depends on the choice of the number w_0. In the theory of differential equations such a singular point is said to be *movable*.

The analytic theory of linear differential equations is in many respects similar to the ordinary theory of linear differential equations in the real domain treated in general courses of analysis (for technical colleges).

In particular, the usual part is played here by the fundamental systems of integrals of the homogeneous linear equation (that is equation (2.109) with $f \equiv 0$). The integrals $w_1(z)$ and $w_2(z)$ of the differential equations

$$w''+f_1(z)w'+f_0(z)w = 0 \tag{2.112}$$

form a fundamental system, if the Wronskian

$$\Delta(z) = \begin{vmatrix} w_1(z) & w_2(z) \\ w_1'(z) & w_2'(z) \end{vmatrix} \tag{2.113}$$

is not identically equal to zero in the domain $\mathscr{E}$. This condition (as also in the real case) is equivalent to the requirement of linear independence of the integrals $w_1(z)$ and $w_2(z)$.

If the Wronskian of two integrals of the differential equation (2.112) *is not identically equal to zero, then it does not become zero at a single point of the domain* $\mathscr{E}$.

In fact, as the functions $w_1(z)$ and $w_2(z)$ satisfy equation (2.112), it follows that

$$\begin{aligned} w_1''+f_1w_1'+f_0w_1 &= 0, \\ w_2''+f_1w_2'+f_0w_2 &= 0. \end{aligned} \tag{2.114}$$

Multiplying the first of these equations by $w_2(z)$, the second by $w_1(z)$ and then subtracting the second result from the first, we find, that

$$(w_1''w_2 - w_2''w_1) + f_1(w_1'w_2 - w_1w_2') = 0. \tag{2.115}$$

The expression, standing in the second bracket, is the Wronskian of the functions $w_1(z)$ and $w_2(z)$. The expression, present in the first bracket, is equal, as a simple calculation shows, to the derivative of this Wronskian. Hence equation (2.115) shows that

$$\Delta' + f_1\Delta = 0,$$

or

$$\mathrm{d}(\mathrm{Ln}\Delta(z)) = -f_1(z)\mathrm{d}z. \tag{2.116}$$

Integrating equation (2.116) along a certain line $l \subset \mathscr{E}$, connecting the point z_0 with a certain point $z \in \mathscr{E}$, we find that

$$\mathrm{Ln}\Delta(z)\Big|_{z_0}^{z} = -\int_{z_0}^{z} f_1(t)\mathrm{d}t,$$

or otherwise

$$\Delta(z) = \Delta(z_0)\exp - \left(\int_{z_0}^{z} f_1(t)\mathrm{d}t\right). \tag{2.117}$$

Here $\Delta(z)$ is the Wronskian of the functions $w_1(z)$ and $w_2(z)$ obtained as a result of the analytic continuation along the curve l of the regular elements of these functions, originally defined in the neighbourhood of the point z_0. The exponential multiplier on the right-hand side of the last equation is different from zero in the domain $\mathscr{E}$ (as there the function $f_1(z)$, and consequently, also the function

$$\int_{z_0}^{z} f_1(t)\mathrm{d}t,$$

is analytic). Hence from the relation (2.117) it follows that:

(*a*) If $\Delta(z_0) = 0$, then $\Delta(z) \equiv 0$ for all $z \in \mathscr{E}$;
(*b*) If $\Delta(z_0) \neq 0$, then $\Delta(z) \neq 0$ for all points $z \in \mathscr{E}$.

From this our statement immediately follows.

Let $w_1(z)$ and $w_2(z)$ be some integrals of equation (2.112), which form a fundamental system. Then the expression

$$w(z) = C_1w_1(z) + C_2w_2(z) \tag{2.118}$$

(where C_1 and C_2 are any complex constants) are general integrals of equation (2.112). This indicates, that whatever the quantities w_0 and w_0' in the initial conditions (2.110) may be, the arbitrary constants C_1 and C_2 can be chosen such that the function $w(z)$, determined by formula (2.118), will satisfy not only equation (2.112) but also the given initial conditions (2.110).

Let us note also, that in the expression (2.118) for the general integral of the differential equation (2.112) the functions $w_1(z)$ and $w_2(z)$ can be replaced by any other integrals $\tilde{w}_1(z)$ and $\tilde{w}_2(z)$ of this equation, which also form a fundamental system, Then, if

$$\tilde{w}_1(z) = h_1w_1(z) + g_1w_2(z),$$

$$\tilde{w}_2(z) = h_2w_1(z) + g_2w_2(z),$$

it follows that

$$\begin{vmatrix} \tilde{w}_1(z) & \tilde{w}_2(z) \\ \tilde{w}_1'(z) & \tilde{w}_2'(z) \end{vmatrix} = \begin{vmatrix} h_1 & g_1 \\ h_2 & g_2 \end{vmatrix} \begin{vmatrix} w_1(z) & w_2(z) \\ w_1'(z) & w_2'(z) \end{vmatrix}. \tag{2.119}$$

In virtue of the relation (2.191) the functions $\tilde{w}_1(z)$ and $\tilde{w}_2(z)$ form a fundamental system if and only if

$$\begin{vmatrix} h_1 & g_1 \\ h_2 & g_2 \end{vmatrix} \neq 0.$$

Finally, we point out the fact that in the complex case also the method of "variation of parameters" which enables us to find the general integral of the differential equation (2.109), when the general integral of the corresponding homogeneous equation (2.112) is known, remains valid.

We will now pass to the study of the properties of the integrals of the differential equation (2.112) in the neighbourhood of an isolated singular point of its coefficients $f_0(z)$ and $f_1(z)$.

Let $z = \gamma$ be such a point. We shall assume that the functions $f_0(z)$ and $f_1(z)$ are regular in the domain U, defined by the inequality

$$0 < |z| < R,$$

and that the point $\zeta \in U$. Also, let the functions $w_1(z)$ and $w_2(z)$

form a fundamental system of solutions of the equation (2.112), defined in the neighbourhood $\mathscr{E}_\zeta$ of the point ζ, and let $\Delta(z)$ be the Wronskian of these functions. Let us continue them analytically along the contour L, which lies in the domain U and passes through the point ζ (the circle $|z-\gamma| = \rho$, where $\rho = |\zeta-\gamma|$, which passes through the point ζ, provides an example of such a contour L). As the result of such a continuation (in which the point z goes round the whole of the contour L in such a way that the domain, bounded by this contour, remains on the left) the integrals $w_1(z)$ and $w_2(z)$, generally speaking, pass into some other integrals $W_1(z)$, $W_2(z)$ of equation (2.112). Let $\tilde{\Delta}(z)$ be the Wronskian of these functions; by virtue of formula (2.117)

$$\tilde{\Delta}(z) = \Delta(\zeta)\exp-(\int_L f_1(t)\mathrm{d}t. \tag{2.120}$$

From this it follows, that the functions $W_1(z)$ and $W_2(z)$ also form a fundamental system in the neighbourhood $\mathscr{E}_\zeta$.

Since the integrals $w_1(z)$ and $w_2(z)$ form a fundamental system, it is possible to select constants a_{ik}, in such a way that we have

$$\begin{aligned} W_1(z) &= a_{11}w_1(z)+a_{12}w_2(z), \\ W_2(z) &= a_{21}w_1(z)+a_{22}w_2(z) \end{aligned} \tag{2.121}$$

(let us emphasize that the values of the constants a_{ik} do not depend on the choice of the contour L, provided that this contour is drawn in accordance with the conditions indicated above). From formulas (2.119) and (2.120) it follows, that

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = \frac{\tilde{\Delta}(\zeta)}{\Delta(z)} = \exp-(\int_L f_1(t)\mathrm{d}t) \neq 0. \tag{2.122}$$

The constants a_{ik} determine by means of formulas (2.121) the changes which take place in the functions $w_1(z)$ and $w_2(z)$ as a result of continuation along the contour L round the point γ. However they depend on the choice of the integrals $w_1(z)$ and $w_2(z)$. In order to remove the influence of this accidental factor, we will replace the integrals $w_1(z)$ and $w_2(z)$ by others, possessing some property, which, as far as possible, defines them uniquely.

Let us search for an integral of equation (2.112) $w = \omega(z)$, *which is*

such that as a result of continuation along the contour L it passes into an integral of equation (2.112)

$$\Omega(z) = s\omega(z), \tag{2.123}$$

where s is a certain complex number.

Every integral $w = \omega(z)$ of equation (2.112) can be represented in the neighbourhood $\mathscr{E}_\zeta$ of the point ζ by the formula

$$\omega(z) = \alpha w_1(z) + \beta w_2(z).$$

By virtue of the relation (2.121) after continuation along the contour L it passes into the integral

$$\Omega(z) = \alpha(a_{11}w_1(z) + a_{12}w_2(z)) + \beta(a_{21}w_1(z) + a_{22}w_2(z)).$$

For equation (2.123) to hold, the quantities α and β must satisfy the equations

$$a_{11}\alpha + a_{21}\beta = s\alpha,$$

$$a_{12}\alpha + a_{22}\beta = s\beta,$$

that is the equations

$$\begin{aligned} (a_{11}s)\alpha + a,_{21}\beta &= 0 \\ a_{12}\alpha + (a_{22}-s)\beta &= 0. \end{aligned} \tag{2.124}$$

As is known from the theory of linear equations, the system (2.124) can be satisfied by values of the quantities α and β, different from zero only in the case when

$$\begin{vmatrix} a_{11}-s & a_{21} \\ a_{12} & a_{22}-s \end{vmatrix}$$

$$= s^2 - (a_{11}+a_{22})s + (a_{11}a_{22} - a_{12}a_{21}) = 0. \tag{2.125}$$

Equation (2.125) is called the *characteristic* equation. Thanks to the relation (2.122) its roots are different from zero.

Later on we shall study two cases separately:

(1) The characteristic equation (2.125) has two distinct, simple roots s_1 and s_2.

(2) The characteristic equation (2.125) has one double root $s_1 = (a_{11}+a_{22})/2$.

Initially let us turn to the first case. In the equations (2.124) let us replace s by s_1; then it reduces to one, for example, if $a_{21} \neq 0$, to the equation

$$(a_{11}-s_1)\alpha + a_{21}\beta = 0. \tag{2.126}$$

The quantities α and β, which satisfy this equation, will automatically also satisfy the second equation of (2.124) (after replacing in it s by s_1). If $a_{21} \neq 0$, we put

$$\alpha = a_{21}, \qquad \beta = s_1 - a_{11}.$$

As a result we find the integral of the equation (2.122)

$$\omega_1(z) = a_{21}w_1(z) + (s_1 - a_{11})w_2(z), \tag{2.127}$$

which passes as a result of continuation along the contour L into the integral $\Omega_1(z) = s_1\omega_1(z)$.

Similarly, replacing in the system (2.124) s by s_2, we find (for $a_{21} \neq 0$) a second integral

$$\omega_2(z) = a_{21}w_1(z) + (s_2 - a_{11})w_2(z), \tag{2.128}$$

which passes as a result of continuation along the contour L into the integral $s_2\omega_2(z)$.

The integrals $\omega_1(z)$ and $\omega_2(z)$ form a fundamental system. This follows from formula (2.119) because of the fact that

$$\begin{vmatrix} a_{21} & s_1 - a_{11} \\ a_{21} & s_2 - a_{11} \end{vmatrix} = a_{21}(s_2 - s_1) \neq 0.$$

If $a_{21} = 0$, then the roots of equation (2.125) are the numbers a_{11} and a_{12} (here we must have $a_{11} \neq a_{22}$). Initially let us put $s = a_{11}$. Then the system (2.124) will be satisfied by the values

$$\alpha = 1, \qquad \beta = \frac{a_{12}}{a_{11} - a_{22}}$$

for $s = a_{22}$ this system is satisfied by the values

$$\alpha = 0, \qquad \beta = 1.$$

Thus, the functions $\omega_1(z)$ and $\omega_2(z)$ are determined in the case selected by the equations

$$\begin{aligned} \omega_1(z) &= w_1(z) + \frac{a_{12}}{a_{11} - a_{22}} w_2(z), \\ \omega_2(z) &= w_2(z). \end{aligned} \tag{2.129}$$

It is obvious, that the functions which we have found constitute a fundamental system.

Now let us turn to the second case. It may happen, that (*a*)

$a_{12} = a_{21}, = 0$, (*b*) $a_{12} = 0, a_{21} \neq 0$; (*c*) $a_{12} \neq 0, a_{21} = 0$; (*d*) $a_{12} \neq 0$, $a_{21} \neq 0$. Let us note, that in cases (*a*), (*b*), and (*c*) $a_{11} = a_{22} = s_1$.

(*a*) If $a_{12} = a_{21} = 0$, the original integrals $w_1(z)$ and $w_2(z)$ possess the required property, whereby as a result of continuation along the contour L they pass into $s_1w_1(z)$ and $s_1w_2(z)$. We will take $w_1(z)$ as $\omega_1(z)$ and $w_2(z)$ as $\omega_2(z)$.

(*b*) If $a_{12} = 0$, $a_{21} \neq 0$, we will take the function $w_1(z)$ as $\omega_1(z)$. It is easy to see, that in this case all the other integrals of equation (2.112), which, when continued along the contour L transform according to formula (2.123), have the form $\alpha\omega_1(z)$. Such an integral does not form together with the function $\omega_1(z)$ a fundamental system.

We will take

$$\omega_3(z) = \frac{s_1}{a_{21}}w_2(z).$$

Then by virtue of the second formula of (2.121)

$$\Omega_2(z) = \frac{a_{22}}{a_{21}}[a_{21}w_1(z) + a_{22}w_2(z)] = s_1[\omega_1(z) + \omega_2(z)].$$

Therefore, in this case we have succeeded in finding integrals $\omega_1(z)$ and $\omega_2(z)$, which constitute a fundamental system and as a result of continuation along the contour L pass into the functions

$$\Omega_1(z) = s_1\omega_1(z), \qquad \Omega_2(z) = s_1[\omega_1(z_1 + \omega_2(z)]. \tag{2.130}$$

(*c*) If $a_{12} \neq 0$, $a_{21} = 0$, we arrive at the same result, by putting

$$\omega_1(z) = w_2(z), \qquad \omega_2(z) = \frac{s_1}{a_{12}}w_1(z).$$

(*d*) If $a_{12} \neq 0$, $a_{21} \neq 0$, proceeding initially just as in the first case we determine the function $\omega_1(z)$ by formula (2.127). It is easy to see, that all the other integrals of equation (2.112) which, when continued along the contour L transform according to formula (2.123), have the form $\alpha\omega_1(z)$. Such integrals do not form together with the function $\omega_1(z)$ a fundamental system. We shall try to choose $\omega_2(z)$ in such a way that the relations (2.130) will hold. For this let us take

$$\omega_2(z) = \alpha w_1(z) + \beta w_2(z).$$

As a result of continuation along the contour L this function will pass into the expression $s_1[\omega_1(z)+\omega_2(z)]$, if

$$\begin{aligned} \alpha(a_{11}-s_1)+\beta a_{21} &= s_1 a_{21}, \\ \alpha a_{12}+\beta(a_{22}-s_1) &= s_1(s_1-a_{11}). \end{aligned} \tag{2.131}$$

The determinant of the system (2.131) is equal to zero (as s_1 is a root of equation (2.125)). Hence values of the coefficients α and β, which simultaneously satisfy both of the equations (2.131), will exist only if the equations

$$\frac{a_{22}-s_1}{a_{21}} = \frac{s_1-a_{11}}{a_{21}} \tag{2.132}$$

are satisfied. In our case the relation (2.132) is satisfied identically (as $s_1 = (a_{11}+a_{22})/2$). Hence it turns out that it is possible to find a function $\omega_2(z)$, satisfying the required property.

We will summarize the results obtained in the following proposition.

THEOREM 7. *Let γ be a singular point of the coefficients of the homogeneous linear differential equation of the second order*

$$w''+f_1(z)w'+f_0(z)w = 0,$$

which remain regular in the domain

$$(U): \qquad 0 < |z-\gamma| < R.$$

Let ζ be a certain point of the domain U, $\mathscr{E}_\zeta$ be a neighbourhood of the point ζ (where $\mathscr{E}: \subset U$), L be a contour, lying in the domain U, which passes through the point ζ and goes once round the point γ.

Then in the neighbourhood $\mathscr{E}_\zeta$ it is always possible to find a fundamental system of integrals $w_1(z)$ and $w_2(z)$ of the given equation, which possess the following property.

As a result of continuation along the contour L (which takes place in such a way that the domain bounded by it remains on the left of the direction of traversal) these integrals are transformed into the functions (cases 1 and 2, (a))

$$\text{(I)} \quad W_1(z) = s_1 w_1(z), \qquad W_2(z) = s_2 w_2(z),$$

or into the functions (cases 2 (a), 2(b), 2(c) and 2(d))

$$\text{(II)} \quad W_1(z) = s_1 w_1(z), \qquad W_2(z) = s_1[w_1(z)+w_2(z)].$$

Here s_1 and s_2 are certain complex constants.†

† The functions $w_1(z)$, $w_2(z)$, $W_1(z)$, $W_2(z)$ in the formulation of theorem 7 correspond to the functions $\omega_1(z)$, $\omega_2(z)$ $\Omega_1(z)$, $\Omega_2(z)$ of our preceding discussion.

Our next problem is the study of the properties of the integrals $w_1(z)$ and $w_2(z)$ in the neighbourhood of the point γ.

Let us note first, that the function $(z-\gamma)^r$ (where r is a certain complex number; here and later on, the branch of this function is taken which is equal to $e^{r\ln(z-\gamma)}$) as a result of continuation along the contour L is transformed into the function $(z-\gamma)^r e^{2\pi i r}$.

Let us now turn to case 1. Taking r first equal to $-(\ln s_1/2\pi i)$, and then equal to $-(\ln s_2/2\pi i)$, we form the expressions

$$(z-\gamma)^{-(\ln s_1/2\pi i)}, \qquad (z-\gamma)^{-(\ln s_2/2\pi i)},$$

which as a result of going round the contour L pass into the quantities

$$\frac{1}{s_1}(z-\gamma)^{-(\ln s_1/2\pi i)}, \quad \frac{1}{s_2}(z-\gamma)^{-(\ln s_2/2\pi i)}.$$

Hence by theorem 7 the products:

$$(z-\gamma)^{-(\ln s_1/2\pi i)}w_1(z), \qquad (z-\gamma)^{-(\ln s_2/2\pi i)}w_2(z) \tag{2.133}$$

are single valued functions in the neighbourhood of the point γ. The point γ will be for them either a point of regularity (we include here the case of a removable singular point), or a pole, or an essential singularity.

If γ is a point of regularity or a pole of the first function of (2.133), then there exists an integer m_1 and a function $\phi_1(z)$ (where $\phi_1(\gamma) \neq 0$) regular at the point γ, such that

$$(z-\gamma)^{-(\ln s_1/2\pi i)}w_1(z) = (z-\gamma)^{m_1}\phi_1(z)$$

(here $m_1 < 0$, if the point $z = \gamma$ is a pole of the function considered; $m_1 > 0$, if this function has a zero at the point $z = \gamma$; $m_1 = 0$, if it is regular and different from zero at the point $z = \gamma$), or otherwise

$$w_1(z) = (z-\gamma)^{r_1}\phi_1(z), \tag{2.134}$$

where

$$r_1 = \frac{\ln s_1}{2\pi i} + m_1.$$

If γ is an essential singularity of the first function of (2.133), let us denote this function by $\phi_1(z)$ and put $r_1 = (\ln s_1/2\pi i)$. As a result we again represent the integral $w_1(z)$ in the form (2.134). However the function $\phi_1(z)$ there will have the point γ as its essential singularity.

Similarly the integral $w_2(z)$ can always be represented in the neighbourhood of the point γ in the form

$$w_2(z) = (z-\gamma)^{r_2}\phi_2(z). \tag{2.135}$$

Here, if γ is a point of regularity or a pole of the second function of (2.133),

$$r_2 = \frac{\ln s_2}{2\pi i} + m_2$$

(where m_2 is determined like the number m_1), and $\phi_2(z)$ is a function, which is regular and not equal to zero at the point $z = \gamma$. If γ is an essential singularity of the second function of (2.133), then it is also an essential singularity of the function $\phi_2(z)$. In this case the quantity $(\ln s_2/2\pi i)$ must be taken as r_2.

Now let us turn to case 2. In the first case it is obvious, that in this case also the integral $w_1(z)$ can be represented in the form (2.134). In order to obtain the representation of the integral $w_2(z)$ let us take into account, that the function $S\ln(z-\gamma)$ (where S is a complex number) is transformed as a result of continuation along the contour L into the function $S\ln(z-\gamma)+2\pi iS$. Hence it is easy to conclude, that subject to our conditions the difference

$$\frac{w_2(z)}{w_1(z)} - \frac{1}{2\pi i}\ln(z-\gamma) \tag{2.136}$$

is a single-valued function in the neighbourhood of the point $z = \gamma$. Then, taking account of the representation (2.134) of the function $w_1(z)$, we find that in the neighbourhood of the point $z = 0$

$$w_2(z) = (z-\gamma)^{r_1}\left[\frac{1}{2\pi i}\phi_1(z)\ln(z-\gamma) + (z-\gamma)^p\phi_3(z)\right]. \tag{2.137}$$

Here the number r_1 and the function $\phi_1(z)$ have the same value, as in formula (2.134); $\phi_3(z)$ is a function, regular in the neighbourhood of the point $z = \gamma$ with the possible exception of the point $z = \gamma$ itself. It may be regular at the point $z = \gamma$ (then the integer p is chosen such that $\phi_3(\gamma) \neq 0$) or have it as an essential singularity (then the number p must be taken to be equal to zero).

Thus we have proved the following proposition.

THEOREM 8. *Let γ be an isolated singular point of the coefficients of the differential equation* (2.112). *Then in the neighbourhood of the*

point γ there exists a fundamental system of integrals of this equation, consisting either of the functions

$$\begin{aligned} w_1(z) &= (z-\gamma)^{r_1}\phi_1(z), \\ w_2(z) &= (z-\gamma_2)^{r_2}\phi_2(z), \end{aligned} \tag{2.138}$$

or of the functions

$$\begin{aligned} w_1(z) &= (z-\gamma)^{r_1}\phi_1(z), \\ w_2(z) &= (z-\gamma)^{r_1}\left[\frac{1}{2\pi i}\phi_1(z)\ln(z-\gamma)+(z-\gamma)^p\phi_3(z)\right]. \end{aligned} \tag{2.139}$$

Here r_1 and r_2 are certain complex constants, p is an integer; $\phi_1(z)$, $\phi_2(z)$ and $\phi_3(z)$ are functions, regular in the neighbourhood of the point γ, with the possible exception of the point γ itself. Each of them is either regular at the point γ (and then different from zero there), or has it as an essential singularity. If γ is an essential singularity of the function $\phi_3(z)$, the number p must be put equal to zero.

15. Regular integrals of a linear differential equation

Let us agree to call the integral $w(z)$ of the equation (2.112) *regular* in the neighbourhood of a certain point γ, if it can be represented there in the form

$$w(z) = (z-\gamma)^r\phi(z) \tag{2.140}$$

or in the form

$$w(z) = (z-\gamma)^r[\phi(z)\ln(z-\gamma)+(z-\gamma)^p\psi(z)], \tag{2.141}$$

where the function $\phi(z)$ or the functions $\phi(z)$ and $\psi(z)$ are regular and different from zero at the point $z = \gamma$, r is a certain complex number, p is a certain integer (positive, negative or zero).

The point γ is said to be a *regular point* of the differential equation (2.112), if in its neighbourhood there exists a fundamental system, consisting of regular integrals of this equation. In particular, if these regular integrals are not regular at the point γ, it is said to be a *regular singular* point of the differential equation (2.112). Our problem is to establish, when the singular point γ of the coefficients of the equation (2.112) is a regular point of the differential equation.

We will take the quantity $z-\gamma$ as a new independent variable. We will denote this new variable by the letter z as before. Thus in

the calculations which follow we have only to consider the case $\gamma = 0$.

Let the integrals of the differential equation (2.112) $w_1(z)$ and $w_2(z)$ form a fundamental system. Then the identities (2.114) and (2.115) hold. For our purpose it is convenient to put the latter in the form

$$f_1(z) = -\frac{w_2''w_1 - w_1''w_2}{w_2'w_1 - w_1'w_2} = -\left[\ln\left(\left(\frac{w_2}{w_1}\right)' w_1^2\right)\right]'.$$

First we will assume, that the integrals $w_1(z)$ and $w_2(z)$ are regular in the neighbourhood of the point $z = 0$ and have there the form (2.138). Then, as a simple calculation shows

$$\left(\frac{w_2}{w_1}\right)' w_1^2 = z^{r_1+r_2-1}[(r_2-r_1)\phi_1\phi_3 + z(\phi_2'\phi_1 - \phi_1'\phi_2)].$$

The expression, which stands in square brackets, becomes $(r_2-r_1)\phi_1(0)\phi_2(0)$ for $z = 0$. The latter quantity, is different from zero if $r_1 \neq r_2$. Hence for $r_1 \neq r_2$

$$f_1(z) = -\left[\ln\left(\left(\frac{w_2}{w_1}\right)' w_1^2\right)\right]' = \frac{1-(r_1+r_2)}{z} + \psi_1(z), \tag{2.142}$$

where $\psi_1(z)$ is a function, regular at the point $z = 0$.

Hence we find, that for $r_1 \neq r_2$

$$f_0(z) = -\frac{w_1'' + f_1 w'}{w_1} = \frac{r_1 r_2}{z^2} + \frac{\alpha}{z} + \psi_0(z), \tag{2.143}$$

where $\psi(z)$ is a function, regular at the point $z = 0$.

If $r_2 = r_1$, then

$$\left(\frac{w_2}{w_1}\right)' w_1^2 = z^{2r_1}(\phi_2'\phi_1 - \phi_1'\phi_2) = z^{2r_1+\nu}\phi(z).$$

Here ν is the order of the zero of the function $\phi_2'\phi_1 - \phi_1'\phi_2$ at the point $z = 0$ (in particular, ν may be equal to zero); $\phi(z)$ is a function, regular and not equal to zero at the point $z = 0$. Hence we find, that even in this case the point $z = 0$ is a pole of not more than the first order of the function $f_1(z)$ and a pole of not more than the second order of the function $f_0(z)$.

Now let us suppose, that in the neighbourhood of the point $z = 0$ the regular integrals $w_1(z)$ and $w_2(z)$ have the form (2.139). Then as a simple calculation shows,

$$\left(\frac{w_2}{w_1}\right)' w_1^2 = z^{2r_1}\chi(z), \tag{2.144}$$

where the function $\chi(z)$ is either regular or has a pole at the point $z = 0$. Hence under any circumstances it follows that

$$f_1(z) = \frac{2r+p}{z} + \chi_1(z).$$

Here p is either the order of the zero of the function $\chi(z)$ at the point $z = 0$ or is the order taken with negative sign of the pole of the function $\chi(z)$ (depending on what, the point $z = 0$ is for the function $\chi(z)$). If $\chi(0) \neq 0, \infty$, then $p = 0$; the function $\chi_1(z)$ is regular at the point $z = 0$.

Hence, in this case also the point $z = 0$ is a pole of not more than the first order for the function $f_1(z)$. Then for the function

$$f_0(z) = -\frac{w_1'' + f_1 w_1'}{w_1}$$

the point $z = 0$ will be a pole of not more than the second order.

Therefore, it has been proved (we are returning to the original independent variable), that in the neighbourhood of the regular point $z = \gamma$ of the differential equation (2.112) the equations

$$\begin{aligned} f_1(z) &= (z-\gamma)^{-1}p(z), \\ f_0(z) &= (z-\gamma)^{-2}q(z) \end{aligned} \tag{2.145}$$

hold, where $p(z)$ and $q(z)$ are regular at the point $z = \gamma$.

We now show, that, conversely, given the relations (2.145) γ is a regular point of the differential equation (2.112). Thus the following theorem is found to be true.

THEOREM 9. *The point $z = \gamma$ will be regular for a certain homogeneous, linear differential equation if and only if, in the neighbourhood of the point γ it has the form*

$$w''(z) + \frac{p(z)}{z-\gamma}w' + \frac{q(z)}{(z-\gamma)^2}w = 0. \tag{2.146}$$

Here $p(z)$ and $q(z)$ are functions, regular at the point $z = \gamma$.

This theorem was established by L. Fuchs a German mathematician of the second half of the nineteenth century.

The necessity of the conditions (2.145) was proved above. Theorem 9 will be completely proved, if we show that equation (2.146) always has two integrals $w_1(z)$ and $w_2(z)$ regular in the neighbourhood of the point $z = \gamma$ which form a fundamental system (later on we again put $\gamma = 0$).

Let us try to satisfy equation (2.146) (where $\gamma = 0$) by the function

$$w(z) = z^{\rho}(c_0+c_1z+c_2z^2+ \dots), \tag{2.147}$$

where the series (2.147) is assumed to be convergent in some neighbourhood of the point $z = 0$, and $c_0 \neq 0$.

In the neighbourhood of the point $z = 0$ let

$$\begin{aligned} p(z) &= a_0+a_1z+a_2z^2+ \dots, \\ q(z) &= b_0+b_1z+b_2z^2+ \dots \end{aligned} \tag{2.148}$$

If (2.147) is a solution of the equation (2.146), substitution of the series (2.147) and (2.148) in the equation (2.146), must result in an identity. Equating to zero the coefficients of the successive powers of z in this identity, we will obtain equations for determining the number γ and the quantities $c_0, c_1, \dots$

Let us equate to zero the coefficients of the ρ-th, lowest power of z in our identity. We obtain the equation

$$[\rho(\rho-1)+a_0\rho+b_0]c_0 = 0. \tag{2.149}$$

but $c_0 \neq 0$; hence equation (2.149) reduces to the equation

$$d(\rho) = \rho(\rho-1)+a_0\rho+b_0 = 0 \tag{2.150}$$

(which is sometimes called *indicial*). In formula (2.147) taking $\rho = \rho_k$, where ρ_k is a root of equation (2.150), we can choose c_0 arbitrarily, since if the function $w(z)$ is an integral of the differential equation (2.146), then so is also every function of the form $cw(z)$. We shall put $c_0 = 1$.

Direct calculation also shows, that, equating to zero the coefficient of z^{q+n} $(n = 1, 2, \dots)$, in our identity, we obtain the equation

$$c_nd(\rho_k+n)+P_{n-1} = 0, \tag{2.151}$$

where P_{n-1} is a certain polynomial in the coefficients $c_1, \dots, c_{n-1}$;

$a_0, \ldots, a_n; b_0, \ldots, b_n$. The coefficient of c_n, the quantity $d(\rho_k+n)$, can be equal to zero only in the case, when one root of equation (2.150) differs from the other by an integer (for example, if $\rho_1 = \rho_2+m$, for $k = 2$ and $n = m$ we obtain as the coefficient of c_m in equation (2.151) the quantity $d(\rho_2+m) = d(\rho_1) = 0$; however in this case $d(\rho_1+n) \neq 0$ for all n). Hence, when the difference $\rho_1-\rho_2$ is not an integer and $\rho_1 \neq \rho_2$, we, putting in equation (2.151) $n = 1, 2, \ldots$, determine from this two sets of values of the coefficients $c_1, c_2, \ldots$. Each of them is uniquely determined by the conditions (2.151).

Thus, for each root ρ of equation (2.150) subject to the condition that the other root $\rho_1 \neq \rho+m$, where m is a positive integer or zero, we obtain one and only one series (2.147), formally satisfying equation (2.146). If both roots ρ_1 and ρ_2 of equation (2.150) satisfy the indicated condition, we obtain two series (2.147), determining in the case of their convergence (which still has to be established) regular integrals of equation (2.146).

Now let us suppose, that $\rho_1 = \rho_2 = r$. It would seem, that in this case equation (2.146) has (if the corresponding series converge) two integrals of the form

$$\begin{aligned} w_1(z) &= z^r(1+c_1z+\ \ldots), \\ w_2(z) &= z^r(1+c_1'z+\ \ldots), \end{aligned} \tag{2.152}$$

where not every c_k is equal to c_k'. However such an assumption is not justified. In fact, if the functions (2.152) are integrals of equation (2.146), then the latter is also satisfied by the sum of the series.

$$w_1(z)-w_2(z) = z^{r'}(h_0+h_1z\ \ldots),$$

where $r' = r+m$ is a positive integer, $h_0 \neq 0$. Then the equation $d(\rho) = 0$ must have, besides the root r, also the root r', differing from r by a positive integer. Thus, if the equation $d(\rho) = 0$ has only one multiple root $\rho_1 = \rho_2$, then the differential equation (2.146) can have only one integral of the form (2.147).

We combine further consideration of the case, where $\rho_1 = \rho_2$, with the consideration of the case, where $\rho_1-\rho_2$ is a positive integer. Therefore, we shall suppose, that $\rho_2 = \rho_1-m$, where m is a positive integer or zero. Then the preceding calculation enables us to find only one series of the form (2.147), formally satisfying the differential equation (2.146). If the indicated series turns out to be convergent,

it defines an integral $w_1(z)$ of equation (2.146). In order to find a second integral of this equation, constituting together with $w_1(z)$ a fundamental system, we substitute in it

$$w(z) = w_1(z)\omega(z). \tag{2.153}$$

As a result instead of the given equation we obtain the equation

$$w_1 z^2 \omega'' + (2w'_1 z^2 + pzw_1)\omega' = 0. \tag{2.154}$$

In order that formula (2.153) should determine a second integral of equation (2.146), any solution of the differential equation (2.154), different from a constant, can be taken as ω. In particular, the function

$$\omega(z) = \int\limits_{\zeta}^{z} \frac{dt}{[w_1(t)]^2} \exp - \left(\int\limits_{\zeta}^{t} \frac{p(\tau)d\tau}{\tau} \right)$$

is a solution (it is easily found by the method of separation of variables from equation (2.154)). It is assumed here that ζ is some point of the domain U; the integrals are taken along lines which lie in this domain; thus $\omega(z)$ will, in general, be a many valued analytic function.

Substituting the expression $\omega(z)$ in formula (2.153), we obtain the function

$$w_2(z) = w_1(z) \int\limits_{\zeta}^{z} \frac{dt}{[w_1(t)]^2} \exp - \left(\int\limits_{\zeta}^{t} \frac{p(\tau)d\tau}{\tau} \right) \tag{2.155}$$

which is the required second integral of equation (2.146).

Let us investigate the function (2.155). From the first equation of (2.148) it follows that

$$\exp - \left(\int\limits_{\zeta}^{t} \frac{p(\tau)d\tau}{\tau} \right) = \exp - \int\limits_{\zeta}^{t} \left(\frac{a_0}{\tau} + a_1 + a_2\tau + \dots \right) d\tau$$

$$= \exp - (a_0 \ln t + a_1 t + \dots - C)$$

$$= t^{-a_0}(-\exp(C - a_1 t - \dots) = t^{-a_0})\phi(t).$$

In these formulas C is a constant, and $\phi(t)$ is a function, regular at the point $t = 0$, and $\phi(0) \neq 0$.

We also find that

$$w_2(z) = w_1(z)\int\limits_{\zeta}^{z}\frac{\mathrm{d}t}{[w_1(t)]^2}\exp-\left(\int\limits_{\zeta}^{t}\frac{p(\tau)\mathrm{d}\tau}{\tau}\right) =$$

$$= w_1(z)\int\limits_{\zeta}^{z}\frac{\phi(t)}{t^{a_0}[w_1(t)]^2}\mathrm{d}t = w_1(z)\int\limits_{\zeta}^{z}t^{-(2\rho_1+a_0)}\frac{\phi(t)}{[1+c_1t+\ \ldots\ ]^2}\mathrm{d}t =$$

$$= w_1(z)\int\limits_{\zeta}^{z}t^{-(2\rho_1+a_0)}\psi(t)\mathrm{d}t.$$

Here $\psi(t)$ is a function, regular, at the point $t = 0$, such that $\psi(0) = g_0 \neq 0$. As ρ_1 and ρ_2 are the roots of equation (2.149), it follows that $\rho_1+\rho_2 = -(a_0-1)$; in addition to this, in our case $\rho_2 = \rho_1-m$. Hence we have

$$2\rho_1+a_0 = m+1$$

a positive integer. Thus, we find, that

$$w_2(z) = w_1(z)\int\limits_{\zeta}^{z}\left(\frac{g_0}{t^{m+1}}+\frac{g_1}{t^m}+\ \ldots\ +\frac{g^m}{t}+g_{m+1}+g_{m+2}t+\ \ldots\ \right)\mathrm{d}t$$

$$= w_1(z)[g_m\ln z+\chi(z)] \tag{2.156}$$

Here $\chi(z)$ is a function, which has a pole of the m-th order at the point $z = 0$. If $m \neq 0$, the coefficient g_m can be equated to zero; then in equation (2.156) the logarithmic term is absent. If $\rho_2 = \rho_1$ and $m = 0$, then as $g_0 \neq 0$, this term must be present in formula (2.156). We obtained already previously the result that for $\rho_2 = \rho_1$ equation (2.146) cannot have two integrals of the form (2.152); but there we used other considerations.

Under all circumstances, if the series (2.147) for the function $w_1(z)$ converges, formula (2.155) determines the second integral $w_2(z)$ of equation (2.146), which forms together with $w_1(z)$ a fundamental set.

Therefore, in both cases (both when $\rho_1-\rho_2 \neq m$, and when $\rho_1-\rho_2 = m$, where m is an integer) we obtain a fundamental system, consisting of regular integrals; the point $z = 0$ is found to be a regular point of the differential equation (2.146).

However, this can be regarded as established only when we have proved the convergence of the resulting series. In the first case it is necessary to prove the convergence of the series (2.147) for $\rho = \rho_1$ and $\rho = \rho_2$ (where ρ_1 and ρ_2 are the roots of the equation $\mathrm{d}(\rho) = 0$, and $\mathrm{d}\ (\rho_1+m) \neq 0$, $\mathrm{d}(\rho_2+m) \neq 0$ for all positive integers m), in the second case for $\rho = \rho_1$ (where $\mathrm{d}(\rho_1) = 0$, but $\mathrm{d}(\rho_1+m) \neq 0$ for all positive integers m). Thus, it is sufficient to show, that the series (2.147) converges, if ρ is a root of the equation $\mathrm{d}(\rho) = 0$ and $\mathrm{d}(\rho+m) \neq 0$ for all positive integers m. In this it has been assumed throughout that the coefficients c_k of the series (2.147) satisfy the relations (2.151).

In equation (2.146) (where $\gamma = 0$) let us put $w = z^{\rho'} v$ (where $\mathrm{d}(\rho') = 0$, $\mathrm{d}(\rho'+m) \neq 0$). By this it is replaced by the equation

$$z^2v'' + zp_1v + q_1v = 0, \tag{2.157}$$

where

$$p_1(z) = p(z) + 2\rho' = (a_0+2\rho)+a_1z+a_2z^2+ \dots,$$
$$q_1(z) = q(z)+\rho'p(z)+\rho'(\rho'-1) =$$
$$= (b_1+\rho'a_1)z+(b_2+\rho'a_2)z^2+ \dots$$

are regular functions at the point $z = 0$ (the absolute term in the series, obtained for the function $q(z)$, is equal to zero, as $\mathrm{d}(\rho') = 0$). Equation (2.147) can be divided by z. As a result we obtain the equation

$$zv'' + p_1v' + q_2 = 0, \tag{2.158}$$

where

$$q_2(z) = (b_1+\rho'a_1)+(b_2+\rho'a_2)z+ \dots$$

We have shown above, that equation (2.146) with $\gamma = 0$ has not more than one solution which can be represented in the neighbourhood of the point $z = 0$ by a series of the form (2.147) with $\rho = \rho_1$, $c_0 = 1$. Hence there can exist only one function $v(z)$, satisfying equation (2.158) which can be represented in the neighbourhood of the point $z = 0$ by a series of the form†

$$v(z) = 1+c_1z+c_2z^2+ \dots). \tag{2.159}$$

† This result does not contradict theorem 4 of the present chapter, as equation (2.158) can be represented in the form

$$v'' = -(p_1v'+q_2)/z$$

where the function on the right-hand side is not, in general, regular for $z = 0$.

If we find by any method a function $v = v(z)$ regular at the point $z = 0$ which satisfies the differential equation (2.158) and the initial condition $v(0) = 1$, then by what has been said it is represented in the neighbourhood of the point $z = 0$ by the series (2.159). By this it is established, that the series (2.159) converges in a certain neighbourhood of the point $z = 0$.

Furthermore, we will put in equation (2.158) $u = (v'/v)$. Then this equation reduces to a system of two differential equations of the first order:

$$z\frac{\mathrm{d}u}{\mathrm{d}z} = -zu^2 - p_1 u - q_2 = F(u, z),$$

$$\frac{\mathrm{d}v}{\mathrm{d}z} = uv. \tag{2.160}$$

First let us find a function $u(z)$, which satisfies the first of these equations and the initial condition

$$u(0) = c_1 = -\frac{b_1 + \rho_1 a_1}{a_0 + 2\rho_1}. \tag{2.161}$$

It is obvious, that the condition $u(0) = c_1$ corresponds to the condition $v(0) = 1$. The value of c_1 given in formula (2.161) is found from equation (2.151) for $\rho = \rho_1$ and $n = 1$. From the relation (2.161)

$$F(c_1, \quad 0) = 0.$$

Hence it follows, that the first equation of the set (2.160) belongs to the type of equation, considered in theorem 6. In order to apply this theorem to our problem, we have only still to convince ourselves of the fact that $F_u'(c_1, 0)$ is not equal to a positive integer.

It is easy to see, that

$$F'u(c_1, \quad 0) = -p_1(0) = -(a_0 + 2\rho_1) = \rho_2 - \rho_1 - 1.$$

If $\rho_2 - \rho_1 - 1 = m$ (where m is a certain positive integer), then $\rho_1 + m + 1 = \rho_2$ and, consequently, is a root of the equation $\mathrm{d}(\rho) = 0$. However such a possibility is excluded by our assumptions.

Therefore, all the conditions for the application of theorem 6 are satisfied. By this theorem there exists (only one) function

$$u(z) = c_1 + d_1 z + d_2 z^2 + \ldots, \tag{2.162}$$

regular at the point $z = 0$ which satisfies the first equation of the

set (2.160) and the initial condition $u(0) = c_1$. Substituting the (convergent) series (2.162) in the second equation of the set (2.160), we give this equation the form

$$\frac{dv}{dz} = v(c_1+d_1z+d_2z^2+\dots). \tag{2.163}$$

We have to find a function $v = v(z)$, regular at the point $z = 0$, which satisfies equation (2.163) and the initial condition $v(0) = 1$. By theorem 2 the given problem has a unique solution. This solution is represented in the neighbourhood of the point $z = 0$ by convergent power series, which by the considerations given above is identical with the series (2.159). Thus we have found, that the latter series is convergent in a certain neighbourhood of the point $z = 0$. This also implies the convergence of the series (2.147) in this neighbourhood.

Hence, theorem 9 is completely proved.

Let us now indicate the conditions which must be satisfied for the integrals of equation (2.112) to remain regular at the point $z = \infty$, and this point to be regular for equation (2.112). Let us put in this equation $z = z_1^{-1}$.

Then we obtain in place of it the equation

$$\frac{d^2w}{dz_1^2}+\left[2z_1^{-1}-z_1^{-2}f_1\left(\frac{1}{z_1}\right)\right]\frac{dw}{dz_1}+z_1^{-4}f_0\left(\frac{1}{z_1}\right)w = 0. \tag{2.164}$$

the integrals of equation (2.112) will be regular at the point $z = \infty$, if the integrals of equation (2.164) are regular at the point $z_1 = 0$. This must be so, if the coefficients of equation (2.164) are regular at the point $z_1 = 0$.

Let us agree to consider the point $z = \infty$ as regular for equation (2.112), *if the point $z_1 = 0$ is regular for equation* (2.164).

By theorem 9, for the point $z_1 = 0$ to be regular for equation (2.164), it is necessary and sufficient, that at this point the functions

$$2-\frac{1}{z_1}f_1\left(\frac{1}{z_1}\right), \qquad \frac{1}{z_1^2}f_0\left(\frac{1}{z_1}\right) \tag{2.165}$$

should be regular. In other words, the function $f_1(1/z_1)$ must have the point $z_1 = 0$ as a zero of not less than the first order, the function $f_0(1/z_1)$ must have the point $z_1 = 0$ as a zero of not less than the second order. If we return to the original independent variable, we can add to theorem 9 the following proposition.

SUPPLEMENT TO THEOREM 9. *For the point* $z = \infty$ *to be regular for the homogeneous linear equation*

$$w'' + f_1 w' + f_0 w = 0,$$

it is necessary and sufficient, that it should be a zero of not less than the first order for the function $f_1(z)$ *and a zero of not less than the second order for the function* $f_0(z)$.

Differential equations of the form (2.112), possessing in the whole of the z-plane only regular singular points, constitute the class of equations of L. Fuchs.

If equation (2.112) belongs to the given class, the function $f_1(z)$ is regular throughout the whole of the z-plane with the possible exception of a finite number of points, at which it has poles of the first order (if this were not so in the z-plane there would be points, in the neighbourhood of which there would occur an infinite number of these poles, that is the function would have non-isolated singular points, which are excluded thanks to our assumptions). Taking into account the regularity of the function $f_1(z)$ at infinity (it has there a zero of not less than the first order), using Liouville's theorem we shall establish, that†

$$f_1(z) = \sum_{k=1}^{n} \frac{A_k}{z - \alpha_k}. \tag{2.166}$$

Here $\alpha_1, \ldots \alpha_n$ are poles of the function $f_1(z)$; $A_1, \ldots, A_n$ are certain complex constants.

Reasoning in a similar way, we find, that in the case considered

$$f_0(z) = \sum_{k=1}^{m} \left[\frac{B_k}{(z-\beta_k)^2} + \frac{C_k}{z-\beta_k}\right], \tag{2.167}$$

where $C_1 + \ldots + C_m = 0$, as we have $\lim\limits_{z\to\infty} z f_0(z) = 0$. Here $\beta_1, \ldots \beta_m$ are poles of the function $f_0(z)$; $B_1, \ldots, B_m$; $C_1, \ldots, C_m$ are certain complex constants.

Therefore, we can formulate the following proposition.

In order that equation (2.112) *should belong to the class of equations of Fuchsian type, it is necessary and sufficient that conditions* (2.166)

† See F.C.V., Chap. VI, Art. 71.

and (2.167) (*including here also the equation* $C_1 + \ldots + C_m = 0$) *should be satisfied.*

16. The Euler–Bessel equation

Let us apply the theory, developed in the preceding article, to the Euler–Bessel differential equation

$$z^2w'' + zw' + (z^2 - \nu^2)w = 0. \tag{2.168}$$

Here ν is a certain complex number. As equation (2.168) contains only the square of the number ν, without losing the generality of our discussion, we can take $\mathrm{Re}\,\nu \geqslant 0$.

The coefficients of equation (2.168) (divided by z^2) are the functions

$$f_1(z) = \frac{1}{z}; \qquad f_0(z) = \frac{z^2 - \nu^2}{z^2} = 1 - \frac{\nu^2}{z^2},$$

regular in the whole of the plane of the complex variable z except for the point $z = 0$. At the point $z = 0$ these functions satisfy the conditions of theorem 9. Consequently, the integrals of the differential equation (2.168) are analytic functions throughout the whole of the (open) z-plane with the possible exception of the point $z = 0$. The point $z = 0$ is a regular point of the differential equation (2.168).

The functions $f_1(z)$ and $f_0(z)$ are regular at the point $z = \infty$; however, as we say at the end of the preceding article, this is not sufficient for the integrals of equation (2.168) to be regular integrals (let alone regular functions) at the point at infinity of the plane. The given integrals will be regular at the point $z = \infty$, if the function $f_1(z)$ has a zero of not less than the first order there, and the function $f_0(z)$ has a zero of not less than the second order there. The function $f_1(z)$ possesses the required property, but the point $z = \infty$ is not a zero of the function $f_0(z)$, since

$$\lim_{z\to\infty} f_0(z) = \lim_{z\to\infty}\left(1 - \frac{\nu^2}{z^2}\right) = 1.$$

Thus, the point $z = \infty$ is not regular for the differential equation (2.168). Sometimes such a singular point (and also in the case of a finite point) and the integrals defined in its neighbourhood are called irregular.

We shall now search for series, representing the integrals of equation (2.168) in the neighbourhood of the point $z = 0$. Comparing the given equation (2.168) with equation (2.146) we find, that in our case $p(z) = 1$, $q(z) = -\nu^2+z^2$, the indicial equation (2.150), has the form

$$d(\rho) = \rho(\rho-1)+\rho-\nu^2 = 0.$$

Hence $\rho_1 = \nu$, $\rho_2 = -\nu$. Thus, if ν is not a non-negative integer (ν cannot be a negative number, as $\mathrm{Re}\nu \geqslant 0$) or half a non-negative odd integer, equation (2.168) has two linearly independent integrals, which are expanded in the neighbourhood of the point $z = 0$ in series of the form (2.147). Using the relations (2.151) to determine the coefficients c_k, we obtain these integrals in the form

$$w_{1,2}(z) = z^\mu \left[1-\frac{(z/2)^2}{1\cdot(\mu+1)}+\frac{(z/2)^4}{1\cdot 2\cdot(\mu+1)(\mu+2)}+\dots \right.$$
$$\left. \dots+(-1)^n\frac{(z/2)^{2n}}{n!(\mu+1)\dots(\mu+n)}+\dots\right]. \qquad (2.169)$$

In order to obtain the integral $w_1(z)$, it is here necessary to take $\mu = \nu$, in order to obtain the integral $w_2(z)$, it is necessary to put $\mu = -\nu$. Theorem 9 ensures the convergence of the series (2.169) in a certain neighbourhood of the point $z = 0$. It is however obvious, that the real domain of convergence of these series is identical with the domains of convergence of the Maclaurin series for the functions $z^{-\nu}w_1(z)$ and $z^\nu w_2(z)$. Each of the latter series converges inside the circle with centre at the point $z = 0$, which passes through the singular point of its sum nearest to the origin. As the functions $z^{-\nu}w_1(z)$ and $z^\nu w_2(z)$ are known to be regular throughout the whole of the (open) z-plane, the given series have an infinite radius of convergence. Consequently, the series (2.169) for $\mu = \nu$ and $\mu = -\nu$ also converge throughout the whole of the (open) z-plane.

In the theory of the integrals of equation (2.166), which are usually called cylinder (Bessel) functions, it is usual instead of $w_1(z)$ and $w_2(z)$ to consider the functions, which differ from them by the constant factor $1/[2^\mu\Gamma(\mu+1)]$, where $\Gamma(\mu+1)$ is Euler's gamma function.† The integrals of equation (2.169) obtained in this way are denoted by the symbols $J_\nu(z)$ and $J_{-\nu}(z)$ and are called

† See F.C.V., Chap. VII, Art. 80.

cylinder functions of the first kind with index (sometimes called order) ν or, respectively, $-\nu$. The series for these functions can be written, using the fact that $\zeta\Gamma(\zeta) = \Gamma(\zeta+1)$, in the form

$$J_\mu(z) = \frac{(z/2)^\mu}{\Gamma(\mu+1)} - \frac{(z/2)^{\mu+2}}{1!\Gamma(\mu+2)} + \dots$$
$$\dots + (-1)^n \frac{(z/2)^{\mu+2n}}{n!\Gamma(\mu+n+1)} + \dots, \tag{2.170}$$

where in order to obtain the function $J_\nu(z)$ it is necessary to put $\mu = \nu$, and in order to obtain the function $J_{-\nu}(z)$ it is necessary to put $\mu = -\nu$. The series (2.170) converge throughout the whole of the (open) z-plane.

Let us now assume, that the parameter ν is equal to half a non-negative odd number, or, in other words, let us put $\nu = k+\frac{1}{2}$, where k is a non-negative integer.

As was established during the proof of theorem 9, in this case also the series (2.170) for $\mu = k+\frac{1}{2}$ determines an integral of equation (2.168)—a cylinder function of the first kind of order (index) $k+\frac{1}{2}$.

It is seen, that under the conditions considered the second integral of equation (2.168)—a cylinder function of the first kind of order $-(k+\frac{1}{2})$—is determined by the expansion (2.170) for $\mu = -(k+\frac{1}{2})$. The series thus obtained also converges throughout the whole of the z-plane. Such a result, although it does not follow from the preceding analysis, does not contradict its results: we have seen, that the function $z^{-\rho_1}w_2(z)$, determined by formula (2.156), can have the point $z = 0$ as a pole or even as a point of regularity (when $g_m = 0$ and the logarithmic term is this formula is absent). We shall not stop for the proof of our assertions, about the series (2.170) for $\mu = -(\nu+\frac{1}{2})$.

It is interesting to note, that the cylinder functions of orders $k+\frac{1}{2}$ and $-(k+\frac{1}{2})$ are expressed in terms of trigonometrical functions. For example, for $\mu = \nu = \frac{1}{2}$ the series (2.170) (taking account of the fact that $\Gamma(\frac{3}{2}) = \sqrt{\pi}/2)$ assumes the form

$$J_{\frac{1}{2}}(z) = \frac{z^{\frac{1}{2}}}{2^{\frac{1}{2}}\Gamma(\frac{3}{2})}\left(1 - \frac{(z/2)^2}{1\cdot(\frac{1}{2}+1)} + \frac{(z/2)^4}{1\cdot 2\cdot(\frac{1}{2}+1)(\frac{1}{2}+2)} - \dots\right)$$
$$= \sqrt{\frac{2}{\pi z}}\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \dots\right) = \sqrt{\frac{2}{\pi z}}\sin z,$$

and for $\mu = \nu = -\frac{1}{2}$ (taking account of the fact that $\Gamma(\frac{1}{2}) = \sqrt{\pi}$) has the form

$$J_{-\frac{1}{2}}(z) = \frac{z^{-\frac{1}{2}}}{2^{-\frac{1}{2}}\Gamma(\frac{1}{2})}\left(1-\frac{(z/2)^2}{1\cdot(-\frac{1}{2}+1)}+\frac{(z/2)^4}{1\cdot 2\cdot(-\frac{1}{2}+1)(-\frac{1}{2}+2)}-\ldots\right)$$
$$= \sqrt{\frac{2}{\pi z}}\left(1-\frac{z^2}{2!}+\frac{z^4}{4!}-\ldots\right) = \sqrt{\frac{2}{\pi z}}\cos z.$$

In the general case (we omit the intermediate steps), it turns out that for $\nu = k+\frac{1}{2}$ and for $\nu = -(k+\frac{1}{2})$

$$J_{k+\frac{1}{2}}(z) = \sqrt{\frac{2}{\pi z}}\left[\sin\left(z-k\frac{\pi}{2}\right)\sum_{n=0}^{[k/2]}\frac{(-1)^n(k+2n)!}{(2n)!(k-2n)!}\frac{1}{(2z)^{2n}}+\right.$$
$$\left.+\cos\left(z-k\frac{\pi}{2}\right)\sum_{n=0}^{[(k-1)/2]}\frac{(-1)^n(k+2n+1)!}{(2n+1)!(k-2n-1)!}\frac{1}{(2z)^{2n+1}}\right], \tag{2.171}$$

$$J_{-(k+\frac{1}{2})}(z) = \sqrt{\frac{2}{\pi z}}\left[\cos\left(z-k\frac{\pi}{2}\right)\sum_{n=0}^{[k-2]}\frac{(-1)^n(k+2n)!}{(2n)!(k-2n)!}\frac{1}{(2z)^{2n}}\right.$$
$$\left.-\sin\left(z-k\frac{\pi}{2}\right)\sum_{n=0}^{[(k-1)/2]}\frac{(-1)^n(k+2n+1)!}{(2n+1)!(k-2n-1)!}\frac{1}{(2z)^{2n+1}}\right]. \tag{2.172}$$

Here the symbols $[k/2]$ and $[(k-1)/2]$ denote respectively, the integral parts of the numbers $k/2$ and $(k-1/)2$.†

Finally, let us suppose, that the parameter ν is a non-negative integer. Then the series (2.170) defines a cylinder function of the first kind of integral order $\nu \geqslant 0$

$$J_\nu(z) = \frac{(z/2)^\nu}{\nu!}-\frac{(z/2)^{\nu+2}}{1!(\nu+1)!}+\ldots+(-1)^n\frac{(z/2)^{\nu+2n}}{n!(\nu+n)!}+\ldots \tag{2.173}$$

(in the deduction of formula (2.173) use is made of the fact that for positive integral q we have $\Gamma(q) = (q-1)!$; here it is considered that $0! = 1$). In particular, for $\nu = 0$ we find, that

$$J_0(z) = 1-\frac{z^2}{2^2}+\ldots+(-1)^n\frac{z^{2n}}{2^2\cdot 4^2\ldots(2n)^2}+\ldots \tag{2.174}$$

† A detailed derivation of these relations can, for example, be found in the book by R. O. Kuzmin, *Bessel Functions*, ONTI, 1935, pp. 57–59.

It is easy to verify, that replacing in formula (2.170) the parameter μ by the number $-\nu$ does not give a new, that is linearly independent of $J_\nu(z)$ integral. In order to determine such a solution of equation (2.163) use must be made of formula (2.155). Carrying out the necessary calculations, we obtain the required second integral $w_2(z)$, which forms together with the function (2.173) a fundamental system. It is obvious that the integral $w_2(z)$ found in this way can be replaced here by any expression of the form $J_\nu(z)+\alpha w_2(z)$, where $\alpha \neq 0$. Using this observation, the indicated second integral is given various forms which are found to be convenient for this or that purpose. It goes without saying, that in all cases we obtain second integrals, possessing the structure, indicated by formula (2.141).

The most frequently used of such second integrals are known as Neumann functions or Weber functions. They are all simultaneously named cylinder functions of the second kind of the corresponding order. The above mentioned Weber functions have the form

$$Y_\nu(z) = \frac{2}{\pi} J_\nu(z)\left(\ln\frac{z}{2} + C\right) - \frac{1}{\pi}\sum_{n=0}^{\nu-1} \frac{(\nu-n-1)!}{n!}\left(\frac{z}{2}\right)^{-\nu-2n}$$

$$-\frac{1}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n (z/2)^{\nu+2n}}{n!\Gamma(\nu+n+1)}\left[\sum_{q=1}^{\nu+n}\frac{1}{q} + \sum_{q=1}^{n}\frac{1}{q}\right],$$

and in the particular case for $\nu = 0$

$$Y_0(z) = \frac{2}{\pi} J_0(z)\left(\ln\frac{z}{2} + C\right) -$$

$$-\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{(n!)^2}\left(\frac{z}{2}\right)^{2n}\left(1+\tfrac{1}{2}+\ldots+\frac{1}{n}\right).$$

Here $C = -\Gamma'(1) = 0.57721566\ldots$ is known as Euler's constant.†

† See the footnote † on page 187.

CHAPTER III

THE LAPLACE TRANSFORMATION AND ITS INVERSION

ONE of the important applications of the theory of analytic functions is the method of integrating linear differential equations (both ordinary and partial), based on the Laplace transformation and its inversion formula. This method enables us not only to obtain the solutions of differential equations, satisfying given initial (and in the case of the mixed problems of mathematical physics, also boundary) conditions in the form of quadratures, expansions in series and so on, but also to find asymptotic expressions and expansions of these solutions for large values of the variables. As the majority of the basic special functions are solutions of linear differential equations, their asymptotic expansions can be obtained from their Laplace transforms by a single method.

The inversion formula of the Laplace transformation is a form of contour integral, and hence the method of which we are speaking, is often called the method of contour integration. It is also closely connected with the so-called operational calculus.

In the present chapter the apparatus of the method of contour integration is developed, that is, definitions are given and the fundamental properties of the Laplace transformation are studied, and also its inversion formula and some theorems which follow from it are derived. As examples simple applications of the results obtained to certain special functions are considered.

17. The primary functions and their Laplace transforms. The uniqueness theorem

We shall consider a function $f(t)$ (generally speaking, complex valued) of the real variable t, which satisfies the following conditions.

1. *$f(t)$ is continuous on the whole of the t-axis with the possible exception of points of discontinuity of the first kind and infinite discontinuities, which are finite in number on any interval of finite length, and $f(t) = 0$ for $t < 0$.*

2. *$f(t)$ is integrable on any interval* $(0, T)$, *where* $T > 0$, *and a number* $s_0 \geqslant 0$ *exists such that*

$$\lim_{T\to\infty}\int_0^T f(t)e^{-s_0 t}dt = \int_0^\infty f(t)e^{-s_0 t}dt$$

exists.†

A function which satisfies conditions 1 and 2, is said to be a *primary function.*

At the points of discontinuity of the first kind, it is in many cases convenient to take as the values of the primary function its limit on the right, that is, to put,

$$f(t) = \lim_{\epsilon\to+0} f(t+\epsilon).$$

In this case it is said, that the primary function is *continuous on the right* (with the possible exception of points of infinite discontinuity).

Let us consider the complex variable

$$p = s+i\sigma, \qquad s = \mathrm{Re}p, \quad \sigma = \mathrm{Im}p.$$

The expression

$$\mathbf{L}f(t) = f^*(p) = \int_0^\infty f(t)e^{-pt}dt \tag{3.1}$$

is called the *Laplace transform or integral* of the primary function $f(t)$.

The Laplace transform, which is, obviously, a function of the complex variable p, will as a rule be denoted by the same letter as the primary function, but with an asterisk. Another notation will be $\mathbf{L}f(t)$.

In order to establish the domain of convergence of the improper integral (3.1) we need the following.

THEOREM 1. *If the integral* (3.1) *converges for* $p = p_0 = s_0+i\sigma_0$, *then it will also converge for all the values of* p, *for which* $s > s_0$.

† At the points of infinite discontinuity $f(t)$ is assumed to be absolutely integrable. The conditions assumed for the primary function can be significantly weakened. However in the present book we are only interested in such a degree of generality, as is sufficient for the applications.

In fact, let us put

$$g(T) = \int_0^T f(t)e^{-p_0 t}\mathrm{d}t.$$

Then by the existence and finiteness of $\lim_{T\to\infty} g(T)$ there exists a constant A, such that $|g(T)| < A$, $0 < T < \infty$. Integrating by parts, we shall have

$$\int_0^T f(t)e^{-pt}\mathrm{d}t = \int_0^T f(t)e^{-p_0 t}e^{-(p-p_0)t}\mathrm{d}t$$

$$= e^{-(p-p_0)t}g(t)\Big|_0^T + (p-p_0)\int_0^T g(t)e^{-(p-p_0)t}\mathrm{d}t$$

$$= e^{-(p-p_0)T}g(T) + (p-p_0)\int_0^T g(t)e^{-(p-p_0)t}\mathrm{d}t. \tag{3.2}$$

But as $I \to \infty$

$$|e^{-(p-p_0)T}g(T)| < Ae^{-(s-s_0)T};$$

the right hand side of the last inequality tends to zero so quickly, that the integral

$$\int_0^\infty g(t)e^{-(p-p_0)t}\mathrm{d}t$$

converges. Consequently the right hand side of equation (3.2) tends to a definite limit as $T \to \infty$, that is, the integral (3.1) converges.

From theorem 1 it follows, that for every primary function $f(t)$ there exists an $s_c \leqslant s_0$, such that its Laplace transform converges for all p, for which $\mathrm{Re}\, p > s_c$, and diverges for all p, for which $Re\, p < s_c$. In the plane of the complex variable p the straight line $s = s_c$ is called the line of convergence, and s_c is the abscissa of convergence of the Laplace integral (3.1).

Thus, the Laplace transform $f^*(p)$ is a function of the complex variable p, defined by the integral (3.1) in the half-plane $\mathrm{Re}\, p > s_c$. The nature of this function is established by the following theorem.

THEOREM 2. *The function* $f^*(p)$ *is regular in the half-plane* Re $p > s_c$.

As a preliminary to the proof of this important theorem we have to establish the following auxiliary proposition.

LEMMA. *The integral* (3.1) *converges uniformly in the closed region* $\bar{D}$ *of the plane of the complex variable* p, *determined by the inequality*

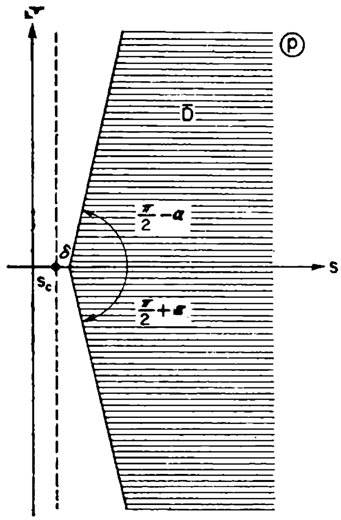

FIG. 15

$|\arg(p-s_c-\delta)| \leqslant \pi/2-\alpha$, where $\delta > 0$ *and* $\alpha > 0$ *are arbitrarily small numbers. This closed region consists of all the points* p, *lying between or on the rays, proceeding from the point* $s_c+\delta$ *of the real axis, and inclined to this axis at the angles* $\pm(\pi/2-\alpha)$ (Fig. 15).

We have to show, that whatever $\epsilon > 0$ may be

$$\left|\int_T^\infty f(t)e^{-pt}\mathrm{d}t\right| < \epsilon \tag{3.3}$$

for all $I > I_0(\epsilon)$ and all p of $(\bar{D})$. As the integral (3.1) converges for $p = s_c+\delta$, putting

$$r(T) \doteq \int_T^\infty f(t)e^{-(s_c+\delta)t}\mathrm{d}t,$$

we shall have, for any $\epsilon' > 0$ that there exists a $T'_0 = T'_0(\epsilon')$, such that $|r(T)| < \epsilon'$ for $T > T'$. But as for any p of $\bar{D}$: $\mathrm{Re}(p-s_c-\delta) > 0$, it follows that

$$\int_T^\infty f(t)e^{-pt}\mathrm{d}t = -\int_T^\infty e^{-(p-s_c-\delta)t}\mathrm{d}r(t)$$

$$= -e^{-(p-s_c-\lambda)T}r(T_1+(p-s_c-\delta)\int_T^\infty e^{(-p-s_c-\delta)t}r(t)\mathrm{d}t$$

and, consequently, for $T > T'_0$

$$\left|\int_T^\infty f(t)e^{-pt}\mathrm{d}t\right| < \epsilon' + |p-s_c-\delta| \int_T^\infty e^{-(s-s_c-\delta)t}\epsilon'\mathrm{d}t$$

$$< \left\{1+\frac{|p-s_c-\delta|}{s-s_c-s}\right\}\epsilon' \leqslant \left(1+\frac{1}{\sin\alpha}\right)\epsilon',$$

as

$$\frac{s-s_c-\delta}{|p-s_c-\delta|} = \cos\arg(p-s_c-\delta) \geqslant \cos\left(\frac{\pi}{2}-\alpha\right) = \sin\alpha.$$

Now, putting $\epsilon' = (\sin\alpha\epsilon)/(1+\sin\alpha)$ and $T'_0(\epsilon') = T_0(\epsilon)$, we obtain the bound (3.3), and the lemma is proved.

In order to prove theorem 2 let us consider any point p of the half-plane $s > s_c$. The quantities δ and α of the preceding lemma can be chosen so that not only the point p considered, but also a sufficiently small circle K_p with centre at this point will belong to the closed region $\bar{D}$. Consequently, in K_p the convergence of the integral (3.1), and hence also of the series

$$\sum_{n=0}^{\infty}\int_n^{n+1} f(t)e^{-pt}\mathrm{d}t \qquad \int_0^\infty f(t)e^{-pt}\mathrm{d}t = f^*(p) \tag{3.4}$$

will be uniform. But every term of this series is an integral function of p,† that is, it is automatically regular in the circle K_p and, consequently, by Weierstrass's theorem,‡ the sum of this series, that is,

† See F.C.V., Chap. VI, Art. 71.

‡ See F.C.V., Chap. VI, Art. 59.

$f^*(p)$, is a function, regular in the circle K_p. Thus it has been proved that $f^*(p)$ is actually regular in the half-plane $s > s_c$.

By the theorem of Weierstrass mentioned above the successive derivatives of the sum of a uniformly convergent series of regular functions can be obtained by term by term differentation of the series, and from the expansion (3.4) we find in this way, that

$$\frac{\mathrm{d}^k f^*}{\mathrm{d}p^k} = \sum_{n=0}^{\infty} \frac{\mathrm{d}^k}{\mathrm{d}p^k} \int_n^{n+1} f(t)e^{-pt}\mathrm{d}t$$

$$= \sum_{n=0}^{\infty} \int_n^{n+1} (-t)^k f(t)e^{-pt}\mathrm{d}t = \int_0^{\infty} (-t)^k f(t)e^{-pt}\mathrm{d}t$$

for all p of the half-plane $s > s_c$ (as each term of the series (3.4) can be differentiated with respect to p any number of times under the integral sign).† This relation shows, that $(-1)^k(\mathrm{d}^k f^*/\mathrm{d}p^k)$ *is the Laplace transform of the primary function* $t^k f(t)$:

$$\mathrm{L}t^k f(t) = (-1)^k \frac{\mathrm{d}^k f^*}{\mathrm{d}p^k} \qquad (\mathrm{Re}p > s_c). \tag{3.5}$$

Let us note also, that the Laplace transform possesses the property of linearity, expressed in the following: *if* $c_1, \ldots, c_n$ *are arbitrary constants, then*

$$\mathrm{L}\{c_1 f_1(t) + \ldots + c_n f_n(t)\} = c_1 \mathrm{L} f_1(t) + \ldots + c_n \mathrm{L} f_n(t). \tag{3.6}$$

This relation is a direct consequence of the elementary properties of definite integrals.

Example 1. Consider the primary function

$$\eta(t) = \begin{cases} 0, & t < 0, \\ 1, & t \geqslant 0. \end{cases}$$

The Laplace transform of this so called *unit step function* will be

$$\mathrm{L}\eta(t) = \eta^*(p) = \int_0^{\infty} e^{-pt}\mathrm{d}t = \frac{1}{p} \qquad (\mathrm{Re}p > 0).$$

† This follows from the fact, that integrals obtained in this way converge uniformly with respect to p in any finite region of the p-plane, as each of them is taken along a finite interval of the t-axis.

Before passing to further examples, let us agree on the following. We shall sometimes write after the symbol **L** of the Laplace transformation an expression for the primary function which is correct only for $t > 0$. Thus, *by* $\mathbf{L}f(t)$ *we shall understand the Laplace transform of the primary function* $\eta(t)f(t)$. In accordance with this in place of $\mathbf{L}\eta(t)$ we shall write $\mathbf{L}1$ and so on.

Example 2. *The Laplace transform of the primary function* $\eta(t)t^\nu$ *for arbitrary* $\nu > -1$. By the relation (3.5) it follows that for $k = 1, 2, \ldots$

$$\mathbf{L}t^k = (-1)^k \frac{d^k \mathbf{L}1}{dp^k} = (-1)^k \frac{d^k(1/p)}{dp^k} = \frac{k!}{p^{k+1}} \qquad (\mathrm{Re}\,p > 0).$$

Let us generalize this relation for an arbitrary real index $\nu > -1$. Let $\mathbf{L}t^\nu = f^*(p)$. Let us first consider real values of $p > 0$. Then,† putting $u = pt$, we find that

$$f^*(p) = \int_0^\infty t^\nu e^{-pt} dt = \frac{1}{p^{\nu+1}} \int_0^\infty u^\nu e^{-u} du = \frac{\Gamma(\nu+1)}{p^{\nu+1}},$$

where by p^α must be understood $e^{\alpha \ln p}$, $\mathrm{Im} \ln p = 0$. But a regular function is completely determined by its values on any segment of the real axis and, consequently, the equation

$$f^*(p) = \frac{\Gamma(\nu+1)}{p^{\nu+1}},$$

must hold, where $p^\alpha = e^{\alpha \ln p}$ and $-\pi/2 < \mathrm{Im} \ln p < \pi/2$. Thus, for any $\nu > -1$

$$\mathbf{L}t^\nu = \frac{\Gamma(\nu+1)}{p^{\nu+1}} \qquad (\mathrm{Re}\,p > 0). \tag{3.7}$$

In particular, it is useful to note, that for $\nu = -\frac{1}{2}$ the following formula is obtained:

$$\mathbf{L}\frac{1}{\sqrt{\pi t}} = \frac{1}{\sqrt{p}} \qquad (\mathrm{Re}\,p > 0, \quad \mathrm{Re}\sqrt{(p)} > 0) \tag{3.7'}$$

(as $\Gamma(\frac{1}{2}) = \sqrt{\pi}$).

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† See F.C.V., Chap. VII, Art. 82.

Example 3. *The Laplace transform of the primary function $\eta(t)\, e^{qt}$, where q is an arbitrary complex number.* It is easy to see, that the Laplace transform of the primary function $\eta(t)e^{qt}$ is equal to

$$\mathbf{L}e^{qt} = \int_0^\infty e^{qt}e^{-pt}\mathrm{d}t = \frac{1}{p-q} \qquad (\mathrm{Re}p > \mathrm{Re}q). \tag{3.8}$$

Putting in this formula $q = i\omega$ and $q = -i\omega$ and using the property of linearity (3.6), we find, that

$$\mathbf{L}\cos\omega t = \mathbf{L}(\tfrac{1}{2}e^{i\omega t}+\tfrac{1}{2}e^{-i\omega t})$$

$$= \tfrac{1}{2}\left(\frac{1}{p-i\omega}+\frac{1}{p+i\omega}\right) = \frac{p}{p^2+\omega^2} \qquad (\mathrm{Re}p > 0); \tag{3.9}$$

$$\mathbf{L}\sin\omega t = \mathbf{L}(\tfrac{1}{2i}e^{i\omega t}-\tfrac{1}{2i}e^{-i\omega t})$$

$$= \tfrac{1}{2i}\left(\frac{1}{p-i\omega}-\frac{1}{p+i\omega}\right) = \frac{\omega}{p^2+\omega^2} \qquad (\mathrm{Re}p > 0). \tag{3.10}$$

By formula (3.5) it follows also from (3.8), that

$$\mathbf{L}t^k e^{qt} = (-1)^k \frac{\mathrm{d}^k}{\mathrm{d}p^k}\frac{1}{p-q} = \frac{k!}{(p-q)^{k+1}} \qquad (\mathrm{Re}p > 0). \tag{3.11}$$

Example 4. *The Laplace transform of the primary function $\eta(t)$ ln t.* In order to calculate the integral

$$\mathbf{L}\ln t = \int_0^\infty \ln t e^{-pt}\mathrm{d}t$$

let us assume, that p is a real positive number, and consider for $\nu \geqslant 0$

$$\int_0^\infty t^\nu e^{-pt}\mathrm{d}t = F(\nu).$$

As was shown in example 2,

$$F(\nu) = \frac{\Gamma(\nu+1)}{p^{\nu+1}}.$$

It is not difficult to show, that $F(\nu)$ can be differentiated with respect to ν under the integral sign for any $\nu \geqslant 0$, that is, that

$$F'(\nu) = \int_0^\infty t^\nu \ln t p e^{-pt} dt$$

and that

$$F'(0) = \int_0^\infty \ln t e^{-pt} dt = \mathbf{L} \ln t.$$

But

$$F'(0) = \frac{\ln p}{p^{\nu+1}} \Gamma(\nu+1)\bigg|_{\nu=0} + \frac{1}{p^{\nu+1}} \Gamma'(\nu+1)\bigg|_{\nu=0}$$

$$= -\frac{1}{p}\ln p + \frac{1}{p}\Gamma'(1).$$

The quantity $-\Gamma'(1)$ is called Euler's constant and is denoted by the letter C (see page 127).†

Therefore,

$$\mathbf{L} \ln t = -\frac{1}{p}\ln p - \frac{C}{p},$$

or as, $C/p = \mathbf{L}C$,

$$\mathbf{L}(\ln t + C) = -\frac{1}{p}\ln p. \tag{3.12}$$

The number of examples of Laplace transforms of concrete primary functions could easily be greatly multiplied.‡ We shall meet many such examples in what follows. But in the examples already given we have discovered the extremely important fact, that *the domain*

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† See F.C.V., pages 315–316. The fact, that Euler's constant C determined there, is equal to $-\Gamma'(1)$, follows from formula (57) on page 315 by differentiation with respect to z and subsequent passage to the limit as $z \to 0$.

‡ See, for example, V. A. Ditkin and P. I. Kuznetsov, *Reference Book on the Operational Calculus* (Gostekhizdat, 1951), where a large number of concrete formulas of this type is given.

of regularity of the Laplace transform $f^*(p)$ *will, in general, be much more extensive than the half-plane* $\operatorname{Re} p > s_c$, *in which it is representable by a Laplace integral.* Thus, for example, the function standing on the right hand side of formula (3.7), has the unique singular point $p = 0$, and the functions standing on the right-hand sides of formulas (3.9) and (3.10), have the singular points $p = \pm i\omega$. As these two latter functions are single-valued, they are regular throughout the whole of the p-plane with the exception of the points $p = \pm i\omega$, at which they have poles of the first order. So far as the Laplace transform (3.7) is concerned, in the case of integral $\nu = 0, 1, \ldots$ it is regular throughout the whole of the p-plane with the exception of the point $p = 0$, at which it has a pole of order $\nu+1$. If, however ν is not integral, then this function is many-valued; the domain of regularity of its single-valued branch, is defined by the condition $-\pi < \operatorname{Im} \ln p \leqslant \pi$, is the p-plane, cut along the real negative semi-axis.

In connection with the fact mentioned above we shall in what follows, without specially mentioning it, *consider all relations between Laplace transforms (such as, for example,* (3.6)), *the statements of theorems* 4 *and* 5 *and so on which follow below as being continued by the method of analytic continuation to the whole of the domain of regularity of the corresponding functions*, in spite of the fact that these relations are established, as a rule, only in the half-plane of convergence of the corresponding Laplace integrals.

To every primary function $f(t)$ there corresponds, obviously, one and only one Laplace transform $f^*(t)$. We shall show, that also conversely, to every $f^*(p)$ there corresponds only one primary function (if at the points of its continuity of the first kind it is taken to be continuous on the right). This proposition forms the content of the following theorem of uniqueness for the primary function.

THEOREM 3. *If* $f^*(p)$ *is the Laplace integral of the primary functions* $f_1(t)$ *and* $f_2(t)$, *which converge in a certain half-plane* $\operatorname{Re} p > s_c$, *then at all the points of continuity of these functions* $f_1(t) = f_2(t)$.

For the proof let us note, that, putting $f_1(t) - f_2(t) = f(t)$ we shall have the equation

$$\int_{\Gamma}^{\infty} f(\tau) e^{-p\tau} d\tau = 0, \tag{3.13}$$

correct in the half-plane $\operatorname{Re} p > s_c$. Let

$$g(t) = \int_0^t f(\tau)e^{-(s_c+1)\tau}d\tau.$$

Then $g(t)$ is continuous on the whole of the t-axis and $g(t) \to 0$ as $t \to \infty$. Putting in (3.13) $p = s_c+1+n(n = 1, 2, \ldots)$ and integrating by parts, we find that

$$g(t)e^{-nt}\Big|_{t=0}^{t=\infty} + n\int_0^\infty g(t)e^{-nt}dt = 0,$$

that is,

$$\int_0^\infty g(t)e^{-nt}dt = 0 \qquad (n = 1, 2, \ldots). \tag{3.14}$$

Let us introduce the new variable u by the equation $e^{-t} = u$, $t = \ln 1/u$, and put $g(t) = h(u)$. Obviously, the function $h(u)$ will be continuous for $0 \leqslant u \leqslant 1$, and $h(0) = h(1) = 0$. The relation (3.14) then passes into the following system of equations:

$$\int_0^1 h(u)u^{n-1}du = 0 \qquad (n = 1, 2, \ldots),$$

from which it follows, that for any polynomial $P(u)$

$$\int_0^1 h(u)P(u)du = 0,$$

or, if $h(u) = h_1(u)+ih_2(u)$, where $h_1(u)$ and $h_2(u)$ are real functions, and $P(u)$ is a real polynomial,

$$\int_0^1 h_1(u)P(u)du = 0, \qquad \int_0^1 h_2(u)P(u)du = 0. \tag{3.15}$$

But if $h_1(u) \not\equiv 0$, then

$$\int_0^1 h_1^2(u)\mathrm{d}u = \eta > 0.$$

Let $|h_1(u)| \leqslant M$, $0 \leqslant u \leqslant 1$. By Weierstrass's theorem on the approximation to continuous functions by polynomials,† there exists a polynomial $P(u)$, such that in the interval $0 \leqslant u \leqslant 1$

$$|h_1(u)-P(u)| < \frac{1}{2M}\eta,$$

that is, $P(u) = h_1(u)+r(u)$, where $|r(u)| < (1/2M)\eta$, $0 \leqslant u \leqslant 1$. From the first relation of (3.15) we now find, that

$$0 = \int_0^1 h_1(u)P(u)\mathrm{d}u = \eta + \int_0^1 h_1(u)r(u)\mathrm{d}u, \tag{3.16}$$

and as

$$\left|\int_0^1 h_1(u)r(u)\mathrm{d}u\right| < M \cdot \frac{1}{2M}\eta = \tfrac{1}{2}\eta,$$

equation (3.16) involves a contradiction, if $\eta > 0$. Consequently, $\eta = 0$, that is $h_1(u) \equiv 0$. It is similarly proved, that $h_2(u) \equiv 0$. Thus, $h(u) \equiv 0$ or $g(t) \equiv 0$. Hence it follows, that at all the points of continuity $f(t) = 0$, that is $f_1(t) = f_2(t)$.

18. The Laplace transforms of integrals and derivatives of primary functions

Let us pass on to theorems which play an important part in applications.

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† See, for example, G. M. Fikhtengol'ts, *Course of the Differential and Integral Calculus*, Vol. III (Gostekhizdat, 1949), page 700.

THEOREM 4. *If* $\mathbf{L}f(t) = f^*(p)$, *then*

$$\mathbf{L}\int^{t} f(u)\mathrm{d}u = \frac{1}{p}\mathbf{L}f(t) = \frac{1}{p} f^*(p).$$

To prove this let us consider the Laplace transform of the function

$$f_1(t) = \int_0^t f(u)\mathrm{d}u,$$

$$\begin{aligned}\mathbf{L}f(t) &= \int_{t=0}^{\infty}\int_{u=0}^{t} f(u)e^{-pt}\mathrm{d}u\mathrm{d}t = \lim_{T\to\infty}\int_{t=0}^{T}\int_{u=0}^{t} f(u)e^{-pt}\mathrm{d}u\mathrm{d}t \\ &= \lim_{T\to\infty}\int_{u=0}^{T}\int_{t=u}^{T} e^{-pt}f(u)\mathrm{d}t\mathrm{d}u = \lim_{T\to\infty}\int_0^T \frac{e^{-pu}e^{-pT}}{p}f(u)\mathrm{d}u.\end{aligned} \tag{3.17}$$

Let the integral (3.1) converge for $\mathrm{Re}\, p = s > s_c$. We shall show that then for $s > s_c$

$$\lim_{T\to\infty}\int_0^T e^{-pT}f(u)\mathrm{d}u = 0. \tag{3.18}$$

For this purpose let us put

$$g(T) = \int_0^T f(u)e^{-p_0u}\mathrm{d}u,$$

where $s > s_0 = \mathrm{Re}\, p_0 > s_c$; then

$$\begin{aligned}\int_0^T e^{-pT}f(u)\mathrm{d}u &= \int_0^T g'(u)e^{-pT+p_0u}\mathrm{d}u \\ &= g(T)e^{-(p-p_0)T} - p_0\int_0^T g(u)e^{-pT+p_0u}\mathrm{d}u,\end{aligned}$$

so that

$$\left|\int_0^T e^{-pT}f(u)\mathrm{d}u\right| \leqslant |g(T)|\,e^{-(s-s_0)T}$$

$$+|p_0|\int_0^T |g(u)|\,e^{-sT+s_0u}\mathrm{d}u. \qquad (3.19)$$

As $g(T)$ tends to a finite limit as $T \to \infty$, it follows that $|g(T)| < A$ for all $I > 0$ where A is a constant. From (3.19) we find, that

$$\left|\int_0^T e^{-pT}f(u)\mathrm{d}u\right| < Ae^{-(s-s_0)T}+|p_0|Ae^{-sT}\int_0^T e^{s_0u}\mathrm{d}u$$

$$< A\{1+|p_0|T\}e^{-(s-s_0)T} \to 0$$

as $I \to \infty$, and equation (3.18) is proved. It now follows from (3.17) that

$$\int_0^\infty f_1(t)e^{-pt}\mathrm{d}t = \frac{1}{p}\int_0^\infty f(u)e^{-pu}\mathrm{d}u,$$

which is the statement of the theorem.

It is easy to generalize this result for repeated integrals. Thus (putting $t_k = t$),

$$\int_0^{t_k}\ldots\int_0^{t_1} f(t_0)\mathrm{d}t_0\ldots\mathrm{d}t_{k-1} = \frac{1}{p^k}\mathrm{L}f(t) \qquad (k = 1, 2, \ldots).$$

This can be proved by repeated application of theorem 4. Let us note, that the relation (3.7) for $\nu = k$ is a consequence of this formula for $f(t) = 1$.

THEOREM 5. *Let* $\mathrm{L}f(t) = f^*(p)$ *and let* $f(t)$ *be continuous and differentiable for all* $t \geqslant 0$, *so that for* $t > 0$

$$f(t)-f(0) = \int_0^t f'(u)\mathrm{d}u, \qquad f(0) = \lim_{\epsilon\to+0} f(\epsilon).$$

Let us put $f'(t) = 0$ for $t < 0$. Then if the function defined in this way is a primary function, we have

$$\mathbf{L}f'(t) = p\mathbf{L}f(t) - f(0) = pf^*(p) - f(0).$$

Let us put $g(t) = f(t) - \eta(t)f(0)$; by (3.6) we have $\mathbf{L}g(t) = f^*p) - - 1/pf(0)$. But

$$g(t) = \int_0^t f'(u)\mathrm{d}u$$

and, consequently, by theorem 4 $1/p\mathbf{L}f'(t) = \mathbf{L}g(t) = f^*(p) - (1/p)f(0)$ whence $\mathbf{L}f'(t) = pf^*(p) - f(0)$.

An especially important case of theorem 5 is that where $f(0) = 0$, that is, *where the primary function is continuous at the point* $t = 0$ (and consequently, on the whole of the axis). *Then*

$$\mathbf{L}f'(t) = p\mathbf{L}f(t), \tag{3.20}$$

if $f'(t)$ is a primary function.

Theorem 5 can be generalized for higher derivatives. Let us introduce the notation:

$$f^{(k)}(0) = f^{(k)}(+0) = \lim_{\epsilon \to +0} f^{(k)}(\epsilon) \qquad (k = 1, 2, \ldots)$$

and suppose that $f(t)$ is differentiable k times for $t > 0$, so that

$$f^{(k-1)}(t) - f^{(k-1)}(0) = \int_0^t f^{(k)}(u)\mathrm{d}u$$

and $f^{(k)}(t) = 0$ for $t < 0$, and also, that $f^{(k)}(t)$ is a primary function. Then

$$\mathbf{L}f^{(k)}(t) = p^k\mathbf{L}f(t) - f(0)p^{k-1} - f'(0)p^{k-2} - \ldots - f^{(k-1)}(0). \tag{3.21}$$

This relation can be proved either by repeated application of theorem 5, or by direct consideration of the primary function

$$g(t) = f(t) - \eta(t)\left\{f(0) + \frac{f'(0)}{1!}t + \ldots + \frac{f^{(k-1)}(0)}{(k-1)!}t^{k-1}\right\},$$

for which

$$\mathbf{L}g(t) = \mathbf{L}f(t) - \frac{f(0)}{p} - \frac{f'(0)}{p^2} - \dots - \frac{f^{(k-1)}(0)}{p^k} = g^*(p),$$

$$g^{(k)}(t) = f^{(k)}(t) \text{ and } g(0) = 0, \qquad g'(0) = 0, \dots, g^{(k-1)}(0) = 0.$$

In fact, by (3.20) $\mathbf{L}g'(t) = pg^*(p)$, $\mathbf{L}g''(t) = p^2g^*(p)$, $\dots$, and, finally, $\mathbf{L}g^{(k)}(t) = \mathbf{L}f^{(k)}(t) = p^kg^*(p)$, which is what it was required to prove.

Remark. In the case of a primary function which is continuous for $t = 0$, theorems 4 and 5 establish a close connexion between the given theory of the Laplace transformation and its inversion by means of contour integration, on the one hand, and what is known as the operational calculus,† on the other. This method consists of the introduction of the operator $p \equiv \mathrm{d}/\mathrm{d}t$ and the determination of rules for working with it. Thus, the symbolic multiplication of the function $f(t)$ by p denotes its differentiation: $pf(t) = f'(t)$. Formally we find from this (assuming that $f(0) = 0$), that

$$f(t) = \int_0^t f'(u)\mathrm{d}u = \frac{1}{p}f'(t),$$

that is, that the symbolic multiplication of the function by $1/p$ denotes its integration between the limits from 0 to t, and so on. The system of rules for working with the operator p, obtained by scientists of the nineteenth century (in particular, Heaviside) without strict foundation, by direct trial and analogy, represents an extremely convenient apparatus for the solution of a series of problems. The strict foundation of this method directly in terms of operators (by operator in modern analysis is understood a law of correspondence, by which each function of one set—the domain of definition of the operator—is put into correspondence with a certain

† Cauchy and a number of other mathematicians of the nineteenth century introduced the notation of differential operator and established simple rules for working with them. M. Vashchenko-Zakharchenko, published in the year 1862 a book, containing an extremely extensive development of the operational method. (M. E. Vashchenko-Zakharchenko, "*The Symbolic Calculus and its Application to the Integration of Linear Differential Equations*", Kiev, 1862). O. Heaviside at the end of the nineteenth century applied the operational method to the solution of a series of electrical problems.

function of another set) is possible in what is known as functional analysis† and was accomplished, mainly, in the works of Soviet mathematicians (A. I. Plesner and others). The theory of the Laplace transformation in the complex domain can also serve as the foundation of the operational method, but without the application of the idea of operator. In this theory, as we have seen, p is not the symbol of an operator, but a complex variable quantity, and *differentiation of the primary functions* (continuous at $t = 0$) *corresponds to multiplication by p of its Laplace transform, and integrating it between the limits from 0 to t to multiplication of its Laplace transform by $1/p$.* The strength of this method consists in the fact that the most important operations of mathematical analysis for applications: the differentiation and integration of functions, is replaced by the simple algebraic operations of multiplication and division of their Laplace transform by the independent variable. Also for the effectiveness of the method it is quite immaterial, in what form it is applied: with operators, applied directly to the primary functions, or with the Laplace transforms of these functions (below we shall have examples of the application of this method in the latter form). The apparatus of the operational method in this latter form with replacement of the Laplace transform by what is known as the "representation" of the function, equal to the Laplace transform, multiplied by p is often called the operational calculus. On the operational calculus there exist many detailed manuals.‡ In them, as a rule, there are given the fundamental theorems of the method of contour integration in terms, free from contour integrals. The distinction of the method of contour integration from the operational calculus as it is usually understood can be seen in the fact that in the solution of problems by the method of contour integration the required primary functions tend to be obtained precisely in the form of a contour integral, which represents the most flexible form of solution, starting with which the behaviour of this solution can be studied from all sides.

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† See, for example, the review article of B. A. Ditkin in *Uzpekhi matem. nauk*, Vol. II (1947), No. 6 (22), pp. 72–158, and also N. I. Akhiezer and I. N. Glazman, *Theory of Linear Operators* (Gostekhizdat, 1950).

‡ See, for example, A. I. Lur'ye, *The Operational Calculus and its Application to Problems of Mechanics*, second edition (Gostekhizdat, 1950), and also M. I. Kontorovich, *The Operational Calculus and Transient Phenomena in Electrical Circuits* (Gostekhizdat, 1949) and A. M. Efros and A. M. Danilevskii, *The Operational Calculus and Contour Integrals* (ONTI, Kharkov, 1937).

19. Limiting relations. The Laplace transforms of the solutions of linear differential equations and some special functions

Let us begin by establishing the connexion, which exists between the limiting values of the primary function and of its Laplace transform, multiplied by p, as $t \to 0$ and $t \to \infty$ and as $p \to \infty$ and $p \to 0$.

THEOREM 6. *Let* $\mathbf{L}f(t) = f^*(p)$, *and assume also, that the primary function* $f(t)$ *satisfies the inequality*

$$|f(t)| < Ae^{s_c t}$$

for all $t > 0$, *where* A *is a certain constant. Then as* p *tends to infinity along the real axis*

$$\lim_{p\to\infty} pf^*(p) = \lim_{t\to+0} f(t) = f(0), \tag{3.22}$$

and if $s_c = 0$ *and there exists* $\lim\limits_{t\to\infty} f(t) = f(\infty)$, *then*

$$\lim_{p\to+0} pf^*(p) = f(\infty). \tag{3.23}$$

Let us suppose that p is real and $p > s_c + \delta$, where $\delta > 0$. Putting $pt = r$, we shall have

$$pf^*(p) = p \int_0^\infty f(t)e^{-pt}\mathrm{d}t = \int_0^\infty f\left(\frac{r}{p}\right)e^{-r}\mathrm{d}r.$$

Then

$$pf^*(p) - f(0) = \int_0^\infty \left\{ f\left(\frac{r}{p}\right) - f(0) \right\} e^{-r}\mathrm{d}r = I_1 + I_2, \tag{3.24}$$

where $(R > 0)$

$$I_1 = \int_0^R \left\{ f\left(\frac{r}{p}\right) - f(0) \right\} e^{-r}\mathrm{d}r,$$

$$I_2 = \int_0^R \left\{ f\left(\frac{r}{p}\right) - f(0) \right\} e^{-r}\mathrm{d}r.$$

Let us begin with the bound for I_2:

$$|I_2| \leqslant \int_R^{\infty} \left| f\left(\frac{r}{p}\right) \right| e^{-r} \mathrm{d}r + |f(0)| e^{-R}$$

but by supposition

$$\left| f\left(\frac{r}{p}\right) \right| < A e^{s_c(r/p)} < A e^{s_c/(s_c+\delta)r}$$

and, consequently,

$$|I_2| < A \int_R^{\infty} e^{-\delta/(s_c+\delta)r} \mathrm{d}r + |f(0)| e^{-R} =$$

$$= \left(1 + \frac{s_c}{\delta}\right) A e^{-\delta/(s_c+\delta)R} + |f(0)| e^{-R}.$$

Let there be given an $\epsilon > 0$. Then R can be chosen so large, that $|I_2| < \frac{1}{2}\epsilon$. Having fixed such a value of R, let us choose P so large that for all $p > P$ and $0 < r < R$

$$\left| f\left(\frac{r}{p}\right) - f(0) \right| < \tfrac{1}{2}\epsilon$$

and

$$|I_1| < \tfrac{1}{2}\epsilon \int_0^R e^{-r} \mathrm{d}r < \tfrac{1}{2}\epsilon.$$

Consequently, for $p > P = P(\epsilon)$ taking account of (3.24), we have

$$|pf^*(p) - f(0)| < \epsilon,$$

which shows that equation (3.22) is true.

From this equation it follows, in particular, that if the primary function $f(t)$ is continuous at the point $t = 0$ and satisfies the conditions of the theorem, then $pf^*(p) \to 0$ as $p \to \infty$.

In order to prove the second assertion (3.32) of the theorem, let us write the equation

$$pf^*(p)-f(\infty) = \int_0^\infty \left\{f\left(\frac{r}{p}\right)-f(\infty)\right\} e^{-r}\mathrm{d}r = I_3+I_4, \tag{3.25}$$

similar to (3.24), where

$$I_3 = \int_0^R \left\{f\left(\frac{r}{p}\right)-f(\infty)\right\} e^{-r}\mathrm{d}r,$$

$$I_4 = \int_R^\infty \left\{f\left(\frac{r}{p}\right)-f(\infty)\right\} e^{-r}\mathrm{d}r.$$

As by the conditions of the given case $s_c = 0$, that is $|f(t)| < A$ for all $t > 0$, it follows that an $R > 0$ can be chosen so small, that we shall have the bound: $|I_3| < \frac{1}{2}\epsilon$. Having fixed such an R, let us choose $P > 0$ so small, that for all $0 < p < P$ and $r > R$

$$\left|f\left(\frac{r}{p}\right)-f(\infty)\right| < \tfrac{1}{2}\epsilon.$$

Then

$$|I_4| < \tfrac{1}{2}\epsilon \int_R^\infty e^{-r}\mathrm{d}r < \tfrac{1}{2}\epsilon,$$

and by (3.25).

$$|pf^*(p)-f(\infty)| < \epsilon$$

for all sufficiently small $p > 0$. This shows that equation (3.23) is true.

The theorem which has just been proved belongs to the number of those which enable us to draw conclusions about the behaviour of a Laplace transform, if the behaviour of the primary function is known. The converse theorem, in which the statement refers to the behaviour of the primary function, presents considerable difficulty. In such theorems (called Tauberian) in addition to the basic premises

on the behaviour of the Laplace transform it is necessary to have a supplementary condition on the primary function itself (one-sided boundedness or a similar condition). Let us give without proof the basic theorem of this type.

If the primary function $f(t)$ *is non-negative,*† $f(t) \geqslant 0$ *and*

$$f^*(p) = \int_0^\infty f(t)e^{-pt}dt$$

converges for $\operatorname{Re} p > 0$*, then if for* p *moving along the real axis*

$$\lim_{p\to\infty} p^\gamma f^*(p) = A, \tag{3.26a}$$

or

$$\lim_{p\to+0} p^\gamma f^*(p) = A, \tag{3.26b}$$

then, correspondingly,

$$\lim_{t\to+0} t^{-\gamma} \int_0^t f(u)du = A, \tag{3.27a}$$

or

$$\lim_{t\to\infty} t^{-\gamma} \int_0^t f(u)du = A. \tag{3.27b}$$

Formulas (3.26) and (3.27) are specially interesting in the case $\nu = 1$. In this case they show that, although by theorem 6 from the existence of

$$\lim_{t\to+0} f(t) = f(0) \quad \text{and} \quad \lim_{t\to\infty} f(t) = f(\infty)$$

there follows the existence, correspondingly, of

$$\lim_{p\to\infty} pf^*(p) \quad \text{and} \quad \lim_{p\to+0} pf^*(p),$$

conversely, from the existence of these latter limits the existence of

† This is the above-mentioned supplementary condition of one-sided boundedness.

the former ones, generally speaking, does not follow, but there follows only the existence, respectively of

$$\lim_{t\to+0}\frac{1}{t}\int_0^t f(u)\mathrm{d}u \quad\text{and}\quad \lim_{t\to\infty}\frac{1}{t}\int_0^t f(u)\mathrm{d}u.$$

It is easy to see this fact from simple examples: if $f(t) = \eta(t)\{1-\cos t\}$ then $f^*(p) = 1/p-p/(1+p^2)$ and

$$\lim_{p\to+0} pf^*(0) = 0, \quad\text{while}\quad \lim_{t\to\infty} f(t),$$

obviously, does not exist.

Let us now consider some simple examples of the preceding results.

Example 1. *The Laplace transforms of the sine integral and cosine integral.* Let

$$f(t) = \eta(t)\int_0^t \frac{\sin u}{u}\mathrm{d}u = \eta(t)\mathrm{Si}t,$$

where

$$\mathrm{Si}t = \int_0^t \frac{\sin u}{u}\mathrm{d}u$$

is what is known as the sine integral. As (see (3.10))

$$\mathbf{L}\sin t = \frac{1}{1+p^2},$$

then, if we put $\mathrm{L}\sin t/t = g^*(p)$, by formula (3.5) for $k = 1$ we shall obtain, that

$$\frac{1}{1+p^2} = -g^{*\prime}(p).$$

Hence it follows, that

$$g^*(p) = C_1 - \int_0^p \frac{\mathrm{d}q}{1+q^2} = C_1 - \mathrm{arctg}\, p.$$

Also, as

$$\lim_{t\to+0}\frac{\sin t}{t} = 1,$$

by the relation (3.22) the equation

$$\lim_{p\to\infty} p(C_1-\mathrm{arctg}\,p) = 1$$

must hold (for p tending to infinity along the real axis), from which it follows, that $C_1 = \pi/2$. Therefore,

$$\mathbf{L}\frac{\sin t}{t} = \frac{\pi}{2} - \mathrm{arctg}\,p = \mathrm{arctg}\frac{1}{p},$$

and consequently,

$$\mathbf{L}\mathrm{Si}\,t = \frac{1}{p}\mathrm{arctg}\,\frac{1}{p}.$$

It is not difficult to show, that

$$\lim_{t\to\infty}\mathrm{Si}\,t$$

exists. From the relation (3.23) we then conclude, that

$$\lim_{t\to\infty}\mathrm{Si}\,t = \int_0^\infty \frac{\sin u}{u}\mathrm{d}u = \lim_{p\to+0}\mathrm{arctg}\frac{1}{p} = \frac{\pi}{2}.$$

Let us now find the Laplace transform of the cosine integral

$$\mathrm{Ci}\,t = -\int_t^\infty \frac{\cos u}{u}\mathrm{d}u.$$

Let p assume real positive values. Then†

$$\mathbf{L}\mathrm{Ci}\,t = -\int_{t=0}^\infty \left\{ \int_{u=t}^\infty \frac{\cos u}{u}\mathrm{d}u \right\} e^{-pt}\mathrm{d}t =$$

† We shall prove neither the permissibility of the change in the order of integration in the following integrals, nor the permissibility of the passage to the limit as $p\to+0$ and of differentiation with respect to p under the integral sign in $\phi(p)$.

$$= -\int_{u=0}^{\infty}\left\{\int_{t=0}^{u} e^{-pt}\mathrm{d}t\right\}\frac{\cos u}{u}\mathrm{d}u =$$

$$= -\int_{u=0}^{\infty}\frac{1-e^{-pu}}{pu}\cos u\mathrm{d}u = -\frac{1}{p}\phi(p),$$

where

$$\phi(p) = \int_{0}^{\infty}\frac{1-e^{-pu}}{u}\cos u\mathrm{d}u.$$

It is not difficult to see that

$$\lim_{p\to+0}\phi(p) = 0 \tag{3.28}$$

and that

$$\phi'(p) = \int_{0}^{\infty} e^{-pu}\cos u\mathrm{d}u = \frac{p}{1+p^2}.$$

Consequently, $\phi(p) = \frac{1}{2}\ \ln(1+p^2)+C_1$, where $C_1 = 0$ by (3.28). Finally we find that

$$\mathbf{L}Cit = -\tfrac{1}{2}p\ln(1+p^2). \tag{3.29}$$

These results, derived for real p, in fact, remain in force for complex p also.

Example 2. *The Laplace transform of the logarithmic integral.* The logarithmic integral is defined by the formula

$$\mathrm{li}t = \int_{0}^{t}\frac{\mathrm{d}u}{\ln u},$$

where if $t > 1$ this integral must be understood as its principal value, that is,

$$\lim_{r\to+0}\left\{\int_{0}^{1-r}\frac{\mathrm{d}u}{\ln u}+\int_{1+\epsilon}^{t}\frac{\mathrm{d}u}{\ln u}\right\}.$$

Let us consider for $t > 0$ the function

$$f(t) = \int_t^{\infty} \frac{e^{-u}}{u} du.$$

Putting $u = -\ln v$, $v = e^{-\alpha}$, we find that

$$f(t) = -\int_0^{e^{-t}} \frac{dv}{\ln v} = -\mathrm{li}\, e^{-t}.$$

Let us calculate the Laplace transform of this function.† Considering p to be real and positive, we have

$$\mathbf{L}f(t) = \int_{t=0}^{\infty} \left\{ \int_{u=t}^{\infty} \frac{e^{-u}}{u} du \right\} e^{-pt} dt =$$

$$= \int_{u=0}^{\infty} \left\{ \int_{t=0}^{u} e^{-pt} dt \right\} \frac{e^{-u}}{u} du = \int_{u=0}^{\infty} \frac{1-e^{-pu}}{pu} e^{-u} du = \frac{1}{p} \phi(p),$$

where

$$\phi(p) = \int_0^{\infty} \frac{1-e^{-pu}}{u} e^{-u} du.$$

But

$$\phi'(p) = \int_0^{\infty} e^{-pu} e^{-u} du = \frac{1}{1+p}.$$

Consequently, $\phi(p) = \ln(1+p) + C_1$, where $C_1 = 0$ by reason of the fact, that

$$\lim_{p \to +0} \phi(p) = 0.$$

† See the footnote on page 150.

Therefore, finally,

$$-\mathbf{L}f(t) = \mathbf{L}\mathrm{li}\,e^{-t} = -\frac{1}{p}\ln(1+p). \tag{3.30}$$

Example 3. *The Laplace transform of a function connected with the probability integral.* By the probability integral the following function is understood:

$$\Phi(t) = \frac{2}{\sqrt{\pi}}\int\limits_0^t e^{-u^2}du.$$

Let us consider the primary function $\eta(t)e^t\Phi(\sqrt{t}) = f(t)$. It is obvious that

$$f'(t) = f(t)+\eta(t)\frac{1}{\sqrt{(\pi t)}} \tag{3.31}$$

and $f(0) = 0$. If $\mathbf{L}f(t) = f^*(p)$, then, it follows that, $\mathbf{L}f'(t) = pf^*(p)$. Thus, passing from equation (3.31) between the primary functions to the equation between their Laplace transforms, we find that by (3.7′)

$$pf^*(p) = f^*(p)+\mathbf{L}\frac{1}{\sqrt{(\pi t)}} = f^*(p)+\frac{1}{\sqrt{p}},$$

where $\mathrm{Re}\sqrt{p} > 0$. Hence

$$\mathbf{L}e^t\Phi(\sqrt{t}) = f^*(p) = \frac{1}{\sqrt{(p)}(p-1)}. \tag{3.32}$$

Example 4. *The Laplace transforms of functions, connected with Fresnel's integrals.* The following functions are called Fresnel's integrals:

$$\mathrm{C}(t) = \frac{2}{\sqrt{\pi}}\int\limits_0 \cos(u^2)du,$$

$$\mathrm{S}(t) = \frac{2}{\sqrt{\pi}}\int\limits_0^t \sin(u^2)du.$$

Let us find the Laplace transforms of the primary functions:

$$f(t) = \eta(t)\frac{1}{\sqrt{\pi}}\int\limits_0^t \frac{\cos(t-\tau)}{\sqrt{\tau}}\mathrm{d}\tau =$$

$$= \eta(t)[\cos t \cdot \mathrm{C}(\sqrt{t}) + \sin t \cdot \mathrm{S}(\sqrt{t})],$$

$$g(t) = \eta(t)\frac{1}{\sqrt{\pi}}\int\limits_0^t \frac{\sin(t-\tau)}{\sqrt{\tau}}\mathrm{d}\tau =$$

$$= \eta(t)[\sin t \cdot \mathrm{C}(\sqrt{t}) - \cos t \cdot \mathrm{S}(\sqrt{t})].$$

Let $\mathbf{L}f(t) = f^*(p)\mathbf{L}g(t) = g^*(p)$. As $f(0) = 0$, $g(0) = 0$ and

$$f'(t) = \eta(t)\frac{1}{\sqrt{(\pi t)}} - g(t),$$

$$g'(t) = f(t),$$

it follows that

$$\mathbf{L}f'(t) = pf^*(p) = \frac{1}{\sqrt{p}} - g^*(p)$$

(where $\mathrm{Re}\,\sqrt{p} > 0$) and

$$\mathbf{L}g'(t) = pg^*(p) = f^*(p).$$

From this it follows, that

$$p^2 f^*(p) = \sqrt{(p)} - f^*(p),$$

that is

$$f^*(p) = \frac{\sqrt{p}}{p^2+1},$$

and consequently,

$$g^*(p) = \frac{1}{\sqrt{(p)}(p^2+1)}.$$

Thus,

$$\mathbf{L}\{\cos t \cdot \mathrm{C}(\sqrt{t}) + \sin t \cdot \mathrm{S}(\sqrt{t})\} = \frac{\sqrt{p}}{p^2+1}, \tag{3.33a}$$

$$\mathbf{L}\{\sin t \cdot \mathrm{C}(\sqrt{t}) - \cos t \cdot \mathrm{S}(\sqrt{t})\} = \frac{1}{\sqrt{(p)}(p^2+1)}. \tag{3.33b}$$

Example 5. *The Laplace transforms of the solutions of linear differential equations with constant coefficients.* The preceding theorems enable us easily to find the Laplace transforms of the solutions of ordinary differential equations with constant coefficients. In fact, let us consider the equation

$$y^{(n)}(t) + a_1 y^{(n-1)}(t) + \ldots + a_n y(t) = f(t), \tag{3.34}$$

where the coefficients $a_1, a_2, \ldots, a_n$ are constants, and the initial conditions: $y(0) = b_0, y'(0) = b_1, \ldots, y^{(n-1)}(0) = b_{n-1}$. It is obvious that if $y(t)$ is any solution of such an equation, then $\eta(t)y(t)$ and $\eta(t)y^{(k)}(t)(k = 1, \ldots, n)$ are primary functions if $\eta(t)f(t)$ is a primary function. Let $\mathbf{L}y(t) = y^*(p)$. Then

$$\mathbf{L}y'(t) = py^*(p) - b_0, \ldots$$

$$\ldots, \mathbf{L}y^{(n)}(t) = p^n y^*(p) - b_0 p^{n-1} - b_1 p^{n-2} - \ldots - b_{n-1}.$$

Also, let $\mathbf{L}f(p) = f^*(p)$. Then, equating to one another the Laplace transforms of the left and right hand sides of equation (3.34), multiplied by $\eta(t)$, we find that

$$\begin{aligned} p^n y^*(p) - b_0 p^{n-1} - \ldots - b_{n-1} + \\ + a_1\{p^{n-1}y^*(p) - b_0 p^{n-2} - \ldots - b_{n-2}\} + \ldots \\ \ldots + a_n y^*(p) = f^*(p), \end{aligned}$$

or

$$Q(p)y^*(p) - H(p) = f^*(p),$$

where

$$Q(p) = p^n + a_1 p^{n-1} + \ldots + a_n$$

is known as the *characteristic polynomial* of equation (3.33), and

$$\begin{aligned} H(p) = b_0 p^{n-1} + (a_1 b_0 + b_1)p^{n-2} + \ldots \\ \ldots + a_{n-1}b_0 + a_{n-2}b_1 + \ldots + b_{n-1}. \end{aligned}$$

Let us note, that the polynomial $H(p)$ becomes identically zero in the case of nul (homogeneous) initial conditions: $b_0 = 0$, $b_1 = 0, \ldots, b_{n-1} = 0$.

Thus,

$$\mathbf{L}y(t) = y^*(p) = \frac{H(p)}{Q(p)} + \frac{f^*(p)}{Q(p)} = y_0^*(p) + y_H^*(p), \tag{3.35}$$

where $y_0^*(p) = H(p)/Q(p)$ is the Laplace transform of the solution of the homogeneous equation (3.35) (for $f(t) = 0$), which satisfies the given (non-homogeneous) initial conditions, and $y_H^*(p) = f^*(h)/Q(p)$ is the Laplace transform of the solution of the non-homogeneous equation (3.34), which satisfies the nul (homogeneous) initial conditions.

Let us note the fact that $y_0^*(p)$ is a *proper rational function of p* (that is the degree of the numerator is smaller than the degree of the denominator).

Similarly it is possible to find the Laplace transforms of the solutions of a set of linear differential equations with constant coefficients.†

Example 6. *The Laplace transforms of the cylinder functions of the first kind.* Formula (3.5) together with formula (3.21) makes it possible in certain cases to find the Laplace transforms of the solutions of linear differential equations of the form

$$M_0(t)y^{(n)}(t) + M_1(t)y^{(n-1)}(t) + \ldots + M_n(t)y(t) = f(t),$$

where $M_0(t), M_1(t), \ldots, M_n(t)$ are polynomials. If $\mathbf{L}y(t) = y^*(p)$, then for $y^*(p)$ there is obtained a differential equation, the order of which is equal to the highest of the degrees of these polynomials. In particular, if all these polynomials are linear functions of t, then $y^*(p)$ satisfies a linear equation of the first order, the general solution of which can always be expressed by quadratures.

Let us apply this general remark to the cylinder functions, which (see § 16) are solutions of the equation

$$y'' + \frac{1}{t}y' + \left(1 - \frac{\nu^2}{t^2}\right)y = 0. \tag{3.36}$$

The cylinder functions of the first kind or order ν are, as we know,

† We shall not stop to go into details on these questions, since these applications of this method are not the purpose of the present treatment. They are discussed in detail in the book A. I. Lur'ye, *The Operational Calculus*, 2nd edition (Gostekhizdat, 1950).

solutions of this equation having the form

$$y = J_\nu(t) = \sum_{n=0}^{\infty} (-1)^n \frac{(t/2)^{2n+\nu}}{n!\Gamma(n+\nu+1)}. \tag{3.37}$$

This series, as is easily verified, converges for all $t > 0$. Let us derive a bound for $J_\nu(t)$ for $t > 0$ and $\nu \geqslant 0$. In the first place

$$|J_\nu(t)| < \sum_{n=0}^{\infty} \frac{(t/2)^{2n+\nu}}{n!\Gamma(n+\nu+1)} < t^\nu \sum_{n=0}^{\infty} \frac{(t/2)^{2n}}{(n!)^2}.$$

Let us put $(t/2)^n = (n!)^2 a_n$. Then

$$\frac{a_{n+1}}{a_n} = \frac{t^2}{4} \frac{1}{(n+1)^2} < \frac{t^2}{(2n+2)(2n+1)} \; (n = 0, 1, \ldots).$$

Consequently,

$$a_1 < a_0 \frac{t^2}{2 \cdot 1}, \quad a_2 < a_1 \frac{t^2}{4 \cdot 3} < a_0 \frac{t^4}{4 \cdot 3 \cdot 2 \cdot 1}, \ldots,$$

and in general

$$a_n < a_0 \frac{t^{2n}}{(2n)!},$$

as $a_0 = 1$, it follows that

$$a_n < \frac{t^{2n}}{(2n)!}.$$

Thus,

$$|J_\nu(t)| < t^\nu \sum_{n=0}^{\infty} \frac{t^{2n}}{(2n)!} < t^\nu \sum_{m=0}^{\infty} \frac{t^m}{m!} = t^\nu e^t.$$

This bound shows, that $\eta(t)J_\nu(t)(\nu = 0, 1, \ldots)$ is a primary function. Similar bounds show, that $\eta(t)J'_\nu(t)$ and $\eta(t)J''_\nu(t)$ are primary functions. Let us find the Laplace transforms of the function $\eta(t)J_\nu(t)$.

Let us first consider $J_0(t)$. Let $\mathbf{L}J_0(t) = J_0{}^*(p)$. As $J_0(0) = 1$ and $J'_0(0) = 0$, it follows that $\mathbf{L}J'_0(t) = pJ_0{}^*(p)-1$, $\mathbf{L}J''_0(t) = p^2J_0{}^*(p)-$ $-p$. Also, $J_0(t)$ satisfies equation (3.36) for $\nu = 0$, that is the identity

$$tJ_0''(t)+J_0'(t)+tJ_0(t) = 0.$$

is satisfied. Consequently, by formula (3.5) we have

$$-\frac{\mathrm{d}}{\mathrm{d}p}\left\{p^2J_0{}^*(p)-p\right\}+pJ_0{}^*(p)-1-\frac{\mathrm{d}}{\mathrm{d}p}J_0{}^*(p) = 0,$$

or

$$\frac{J_0{}^{*\prime}(p)}{J_0{}^*(p)} = -\frac{p}{1+p^2},$$

whence

$$J_0{}^*(p) = \frac{C_1}{\sqrt{(1+p^2)}},$$

where that branch of the square root is taken, for which $\sqrt{1} = 1$. In order to determine the arbitrary constant C_1 let us use the relation (3.22), by virtue of which

$$\lim_{p\to\infty} pJ_0{}^*(p) = J_0(0) = 1.$$

Consequently, $C_1 = 1$ and

$$\mathbf{L}J_0(t) = -\frac{1}{\sqrt{(1+p^2)}}.$$

The Laplace transforms of the functions $\eta(t)J_\nu(t)$ for $\nu = 1, 2, \ldots$ are most easily obtained, if use is made of certain simple relations, which follow from the expansion (3.37). In the first place, from (3.37) it easily follows that $J_1(t) = -J'_0(t)$, so that if $\mathbf{L}J_1(t) = J_1{}^*(p)$ then

$$J_1{}^*(p) = -pJ_0{}^*(p)+1 = -\frac{p}{\sqrt{(1+p^2)}}+1 = \frac{\sqrt{(1+p^2)}-p}{\sqrt{(1+p^2)}}.$$

Also, let us, again using the expansion (3.37), form the difference

$$J_{\nu-1}(t)-2J_\nu{}'(t) =$$

$$= \sum_{n=0}^{\infty}(-1)^n\frac{(t/2)^{2n+\nu-1}}{n!\Gamma(n+\nu)}-2\sum_{n=0}^{\infty}(-1)^n\frac{\frac{1}{2}(2n+\nu)(t/2)^{2n+\nu-1}}{n!\Gamma(n+\nu+1)} =$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{(t/2)^{2n+\nu-1}}{n!\Gamma(n+\nu)} \left(1 - \frac{2n+\nu}{n+\nu}\right) =$$

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{(t/2)^{2n+\nu-1}}{(n-1)!\Gamma(n+\nu+1)}$$

(the last equation is true in virtue of the fact that the term which corresponds to $\eta = 0$ is equal to zero). In the last sum let us replace the index of summation n by $n+1$. We then find, that

$$J_{\nu-1}(t) - 2J_\nu'(t) = \sum_{n=0}^{\infty} (-1)^n \frac{(t/2)^{2n+\nu+1}}{n!\Gamma(n+\nu+2)} = J_{\nu+1}(t). \tag{3.38}$$

Let us now prove the following general formula:

$$\mathbf{L}J_\nu(t) = \frac{[\sqrt{(1+p^2)}-p]^\nu}{\sqrt{(1+p^2)}}. \tag{3.39}$$

The proof can be carried out by the method of mathematical induction applied to the relation (3.38). As we have seen, formula (3.39) is true for $\nu = 0$ and $\nu = 1$. Let us assume, that it is true for $\nu = k-1$ and $\nu = k (k \geqslant 1)$, and prove, that it is then true for $\nu = k+1$. Therefore, let

$$\mathbf{L}J_k(t) = \frac{[\sqrt{(1+p^2)}-p]^k}{\sqrt{(1+p^2)}}, \quad \mathbf{L}J_{k-1}(t) = \frac{[\sqrt{(1+p)}-p]^{k-1}}{\sqrt{(1+p^2)}}.$$

Then by (3.38)

$$\mathbf{L}J_{k+1}(t) = \frac{[\sqrt{(1+p^2)}-p]^{k-1}}{\sqrt{(1+p^2)}} - 2p\frac{[\sqrt{(1+p^2)}-p]^k}{\sqrt{(1+p^2)}} =$$

$$= \frac{[\sqrt{(1+p^2)}-p]^{k-1}}{\sqrt{(1+p^2)}} \left\{ 1 - 2p\sqrt{(1+p^2)} + 2p^2 \right\} =$$

$$= \frac{[\sqrt{(1+p^2)}-p]^{k-1}}{\sqrt{(1+p^2)}} [\sqrt{(1+p^2)}-p]^2 = \frac{[\sqrt{(1+p^2)}-p]^{k+1}}{\sqrt{(1+p^2)}}$$

which is what it was required to prove.

20. The inversion formula of the Laplace transformation

Let us now pass to the derivation of the fundamental formula of the method of contour integration—the inversion formula of the Laplace transformation.

THEOREM 7. *Let the function* $f^*(p)$ *be regular in the half-plane* $\operatorname{Re} p > s_0 > 0$ and

1. $|f^*(p)| < A < \infty$ *in any half plane* $\operatorname{Re} p \geqslant s_0' > s_0$.
2. *The improper integral*

$$\int_{-\infty}^{\infty} f^*(s+i\sigma)e^{i\sigma t}\mathrm{d}\sigma$$

converge for all $s > s_0$ *and be a continuous function of* t *for* $t > 0$.

3. *for all* $s > s_0$

$$\int_{-\infty}^{\infty} \left|\frac{f^*(s+i\sigma)}{s+i\sigma}\right| \mathrm{d}\sigma < B < \infty.$$

Then

$$f^*(p) = \mathbf{L}f(t),$$

where for $t > 0$

$$f(t) = \frac{1}{2\pi i}\int_L f^*(p)e^{pt}\mathrm{d}p, \tag{3.40}$$

and the path L *is the straight line* $\operatorname{Re} p = s > s_0$, *oriented in the direction of increase of* $\operatorname{Im} p = \sigma$.

Proof. Let us first note, that by the uniqueness theorem of the primary function (theorem 3, Art. 17) other primary functions of $f^*(p)$ which are continuous for $t > 0$ cannot exist.

In fact, formula (3.40) can be written in the form

$$f(t) = \frac{1}{2\pi i}e^{\sigma t}\int_{-\infty}^{\infty} f^*(s+i\sigma)e^{i\sigma t}\mathrm{d}\sigma.$$

In virtue of condition 2, $f(t)$ is a continuous function of t for all $t > 0$.

Let us now show, that for $s > s_0$

$$\lim_{\substack{\epsilon \to +0 \\ T \to \infty}} \int_{\epsilon}^{T} f(t)e^{-pt}\mathrm{d}t$$

exists and is equal to $f^*(p)$ (we have to begin by integrating from $\epsilon > 0$ to T, as the Laplace integral of the primary function (3.40) may also be improper at its lower limit). For this purpose, having fixed $p = s+i\sigma$, let us consider the function

$$F(p;\epsilon, T) = \int_{\epsilon}^{T} f(t)e^{-pt}\mathrm{d}t = \int_{\epsilon}^{T} \left\{ \frac{1}{2\mu i} \int_{L^*} f^*(p^*)e^{p^*t}\mathrm{d}p^* \right\} e^{-pt}\mathrm{d}t,$$

where $p^* = s^*+i\sigma^*$ is a variable point of the path $L^*:s^* =$ constant $(\infty < \sigma^* < \infty)$, and $s_0 < s^* < s$.

Let us consider the segment $L_p^*(-b \leqslant \sigma^* \leqslant b)$ of the path L^* and put

$$\frac{1}{2\pi}\int_{-\infty}^{-b} f^*(s^*+i\sigma^*)e^{(s^*+i\sigma^*)t}\mathrm{d}\sigma^* = \theta_b(t),$$

$$\frac{1}{2\pi}\int_{h}^{\infty} f^*(s^*+i\sigma^*)e^{(s^*+i\sigma^*)t}\mathrm{d}\sigma^* = \theta_b(t).$$

By virtue of condition 2

$$\lim_{b\to\infty} \theta_b(t) = \lim_{b\to\infty} \theta_b(t) = 0 \qquad (\epsilon \leqslant t \leqslant T). \tag{3.41}$$

On the other hand,

$$F(p;\epsilon,T) = \int_{\epsilon}^{T} \left\{ \theta_b(t) + \frac{1}{2\pi i}\int_{L_b^*} f^*(p^*)e^{p^*t}\mathrm{d}p^* + \theta_b(t) \right\} e^{-pt}\mathrm{d}t =$$

$$= \int_{\epsilon}^{T} \theta_b(t)e^{-pt}\mathrm{d}t +$$

$$+\frac{1}{2\pi i}\int_{L_a^*} f^*(p^*)\left\{ \int_{\epsilon}^{T} e^{-(p-p^*)t}\mathrm{d}t \right\} \mathrm{d}p^* + \int_{\epsilon}^{T} \theta_b(t)e^{-pt}\mathrm{d}t, \tag{3.42}$$

where the inversion of the order of integration performed in the middle term is justified by virtue of the finiteness of the intervals of integration and the continuity of the integrand. Passing in formula (3.42) to the limit as $b \to \infty$ and taking account of the relation (3.41), we find that

$$F(p;\epsilon,T) = \frac{1}{2\pi i}\int_{L^*} f^*(p^*)\left\{\int_{\epsilon}^{T} e^{-(p-p^*)t}\mathrm{d}t\right\}\mathrm{d}p^* =$$

$$= \frac{1}{2\pi i}\int_{L^*} f^*(p^*)\frac{e^{-(p-p^*)\epsilon}-e^{-(p-p^*)T}}{p-p^*}\mathrm{d}p^* = \frac{1}{2\pi i}\int_{L^*}\frac{f^*(p^*)}{p-p^*}\mathrm{d}p^* -$$

$$-\frac{1}{2\pi i}\int_{L^*}\frac{f^*(p^*)}{p-p^*}\{1-e^{-(p-p^*)\epsilon}-e^{-(p-p^*)T}\}\mathrm{d}p^*. \tag{3.43}$$

But

$$\left|\int_{L^*}\frac{f^*(p^*)}{p-p^*}\{1-e^{-(p-p^*)\epsilon}-e^{-(p-p^*)T}\}\mathrm{d}p^*\right| <$$

$$< \int_{L^*}\left|\frac{f^*(p^*)}{p-p^*}\right| |1-e^{-(p-p^*)\epsilon}|\, |\mathrm{d}p^*| +$$

$$+ e^{-(s-s^*)T}\int_{L^*}\left|\frac{f^*(p^*)}{p-p^*}\right| |\mathrm{d}p^*|. \tag{3.44}$$

It is not difficult to see, that there exists a quantity $c < \infty$, not depending on σ^*, such that†

$$\frac{1}{|p-p^*|} \leqslant \frac{c}{|p^*|}.$$

† In fact, it is possible to put

$$c = \max_{L^*}\left|\frac{p^*}{p-p^*}\right| = \max_{-\infty<\ *<}\sqrt{\frac{s^{*2}+\sigma^{*2}}{(s-s^*)^2+(\sigma-\sigma^*)^2}}.$$

The quantity c, of course, depends both on the position of the point p to th right of the path L^* (that is on s and σ), and on the abscissa s^* of the path L

Consequently, by condition 3

$$\int_{L^*} \left|\frac{f^*(p^*)}{p-p^*}\right| |\mathrm{d}p^*| \leqslant c \int_{L^*} \left|\frac{f^*(p^*)}{p^*}\right| |\mathrm{d}p^*| < cB < \infty, \tag{3.45}$$

in consequence of which the second term on the right-hand side of the inequality (3.44) tends to zero as $T \to \infty$. We shall show, that the first term also tends to zero as $\epsilon \to +0$. By virtue of the convergence of the integral (3.45) it is possible to choose b so large, that the inequalities

$$\int_{-\infty}^{-b} \left|\frac{f^*(p^*)}{p-p^*}\right| \mathrm{d}\sigma^* < \tfrac{1}{6}\eta, \qquad \int_{b}^{\infty} \left|\frac{f^*(p^*)}{p-p^*}\right| \mathrm{d}\sigma^* < \tfrac{1}{6}\eta,$$

are satisfied where $\eta > 0$ is any arbitrarily small number. Having fixed b in this way, let us choose ϵ so small, that the inequality $|1-e^{-(p-p^*)\epsilon}| < (1/3B)\eta$ is satisfied for $-b \leqslant \sigma^* = \operatorname{Im} p^* \leqslant b$. Then, as we always have

$$|1-e^{-(p-p^*)\epsilon}| < 2,$$

$$\int_{L^*} \left|\frac{f^*(p^*)}{p-p^*}\right| |1-e^{-(p-p^*)\epsilon}|\, |\mathrm{d}p^*| <$$

$$< 2\int_{-\infty}^{-b} \left|\frac{f^*(p^*)}{p-p^*}\right| \mathrm{d}\sigma^* + \frac{1}{3B}\eta \int_{Lb^*} \left|\frac{f^*(p^*)}{p-p^*}\right| |\mathrm{d}p^*| +$$

$$+2\int_{b}^{\infty} \left|\frac{f^*(p^*)}{p-p^*}\right| \mathrm{d}\sigma^* < \frac{1}{3}\eta + \frac{1}{3B}\eta \,.\, B + \frac{1}{3}\eta = \eta$$

and, consequently,

$$\lim_{\epsilon\to+0} \int_{L^*} \left|\frac{f^*(p^*)}{p-p^*}\right| |1-e^{-(p-p^*)\epsilon}|\, |\mathrm{d}p^*| = 0.$$

By (3.43) we now find, that

$$\lim_{\substack{\epsilon\to+0\\ T\to\infty}} F(p;\epsilon, T) = \frac{1}{2\pi i}\int_{L^*} \frac{f^*(p^*)}{p-p^*}\mathrm{d}p^* = F_1(p). \tag{3.46}$$

It remains to prove, that $F_1(p) = f^*(p)$. Let us consider the contour Q_l, which is a rectangle with vertices at the points s^*-ib, s^*+ib, $S+ib$, $S-ib$, where $S > s$ and $b > |\sigma|$ (Fig. 16). If the contour Q_b is oriented in such a way that its side L_b^* runs in the direction of increase of σ^* then by Cauchy's theorem

$$\frac{1}{2\pi i}\oint\limits_{Q_b}\frac{f^*(p^*)}{p-p^*}\mathrm{d}p^* = -\frac{1}{2\pi i}\oint\limits_{Q_b}\frac{f^*(p^*)}{p^*-p}\mathrm{d}p^* = f^*(p).$$

Consequently, by (3.46)

$$F_1(p) = f^*(p) - \lim_{b\to\infty}\frac{1}{2\pi i}(I_1+I_2+I_3), \tag{3.47}$$

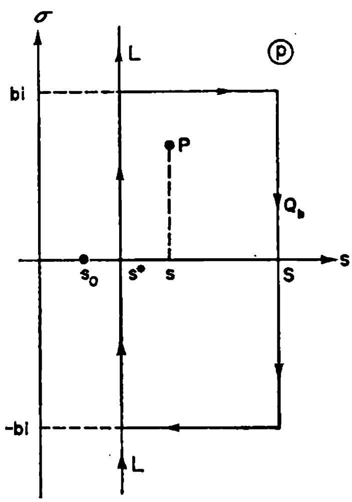


FIG. 16

where

$$I_1 = \int\limits_{\lambda=s^*}^{S}\frac{f^*(\lambda+ib)}{\lambda-s+i(b-\sigma)}\mathrm{d}\lambda,$$

$$I_2 = i\int\limits_{u=b}^{-b}\frac{f^*(S+i\mu)}{S-s+i(\mu-\sigma)}\mathrm{d}\mu,$$

$$I_3 = \int\limits_{\lambda=S}^{s^*}\frac{f^*(\lambda-ib)}{\lambda-s+i(b-\sigma)}\mathrm{d}\lambda.$$

It is not difficult to see, that by virtue of condition 1

$$\lim_{b\to\infty} I_1 = \lim_{b\to\infty} I_3 = 0$$

passing in (3.47) to the limit as $b \to \infty$, we hence find that

$$F_1(p) = f^*(p) - \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{f^*(S+i\mu)}{S-s+i(\mu-\sigma)}\,d\mu. \tag{3.48}$$

From condition 3 it follows, that the last integral is absolutely convergent. In fact, as for fixed $S > s$ and σ variable μ†

$$\frac{1}{|S-s+i(\mu-\sigma)|} \leqslant \frac{c}{|S+i\mu|},$$

where c does not depend on μ, it follows that

$$\int_{-\infty}^{\infty}\left|\frac{f^*(S+i\mu)}{S-s+i(\mu-\sigma)}\right| d\mu < c\int_{-\infty}^{\infty}\left|\frac{f^*(S+i\mu)}{S+i\mu}\right| d\mu < cB.$$

Let us show, that this integral tends to zero as $S\to\infty$. For this it is obviously sufficient, to show that

$$\lim_{S\to\infty}\int_{-\infty}^{\infty}\left|\frac{f^*(S+i\mu)}{S+i\mu}\right| d\mu = 0. \tag{3.49}$$

Let $\eta > 0$ be given. Let us choose b so large, that for all $S > s$ the inequalities

$$\int_{-\infty}^{-b}\left|\frac{f^*(S+i\mu)}{S+i\mu}\right| d\mu < \tfrac{1}{3}\eta, \qquad \int_{b}^{\infty}\left|\frac{f^*(S+i\mu)}{S+i\mu}\right| d\mu < \tfrac{1}{3}\eta$$

are satisfied. Having fixed b in this way, let us choose S so large that the inequality

$$\left|\frac{f^*(S+i\mu)}{S+i\mu}\right| < \frac{1}{6b}\eta$$

† See the footnote on page 162. In the given case of course, c depends on s, σ and S.

is satisfied. On the basis of condition 1, this is obviously possible. Then

$$\int\limits_{-\infty}^{\infty}\left|\frac{f^*(S+i\mu)}{S+i\mu}\right|d\mu =$$

$$= \int\limits_{-\infty}^{-b}\left|\frac{f^*(S+i\mu)}{S+i\mu}\right|d\mu + \int\limits_{-b}^{b}\left|\frac{f^*(S+i\mu)}{S+i\mu}\right|d\mu +$$

$$+ \int\limits_{b}^{\infty}\left|\frac{f^*(S+i\mu)}{S+i\mu}\right|d\mu < \frac{1}{3}\eta + \frac{1}{6b}\eta \cdot 2b + \frac{1}{3}\eta = \eta.$$

This also shows that the relation (3.49) is true. Passing, finally, in formula (3.48) to the limit as $S \to \infty$, we find, that $F_1(p) = f^*(p)$ and the theorem is proved.†

Let us make some remarks on theorem 7.

Remark 1. This theorem, of course, does not contain the assertion, that conditions 1, 2 and 3 are necessary for the function $f^*(p)$ to be the Laplace transform of some primary function $f(t)$. We have only proved, that the conditions stated are sufficient for this. Thus, if in the inversion formula (3.40) there is substituted in place of $f^*(p)$ the Laplace transform of any primary function $f(t)$, where this Laplace transform does not satisfy conditions 1, 2 and 3 (that is at least one of them), then the right hand side of this formula may both represent, and not represent a primary function. What the position will be in such cases can in fact only be established by finer considerations.

Remark 2. Formula (3.40) can be proved also for other sufficient conditions on the function $f^*(p)$. The conditions 1, 2 and 3 taken by us are the most convenient for those applications, which we have in view. Let us note, that condition 2 will be automatically satisfied, if

$$\int\limits_{-\infty}^{\infty}|f^*(s+i\sigma)|d\sigma < \infty$$

† This discussion shows, however, that the integral in formula (3.48) is equal to zero for all $S > s$.

for every $s > s_0$, or what is a weaker condition, than the preceding, if

$$\int_{-\infty}^{\infty} f^*(s+i\sigma)e^{i\sigma t}d\sigma$$

converges uniformly with respect to t for every $s > s_0$.

Remark 3. From the proof of the theorem it follows, that if the conditions of the theorem are satisfied, the integral (3.38) does not depend on the abscissa of the path L, which is traversed in the half-plane of regularity of $f^*(p)$.

Remark 4. If in place of condition 1 it is required, that

$$f^*(p) \to 0 \quad \text{and} \quad |p| \to \infty, \quad \mathrm{Re}p \geqslant s_0' > s_0, \tag{3.50}$$

then the inversion formula (3.40) remains in force also for $t < 0$, that is reduces for negative t to the equation $f(t) = 0$ (this sometimes takes place also when the condition (3.50) is not satisfied).

In fact, let $t < 0$. Denoting the segment $-b \leqslant \sigma \leqslant b$ of the path L by L_b, we shall have

$$f(t) = \lim_{b\to\infty} \frac{1}{2\pi i} \int_{L_b} f^*(p)e^{pt}dp.$$

Let us connect the ends of the segment L_b by the arc K_b of the circle with centre at the origin and of radius $R = \sqrt{(s^2+b^2)}$ (Fig. 17), situated in the half plane Re $p > s$. Also let us denote by C_b the contour, consisting of L_b and K'_b, where K'_b is the path K_b, traversed in the opposite direction. Then by Cauchy's theorem

$$\oint_{C_b} f^*(p)e^{pt}\mathrm{dp} = 0.$$

Consequently,

$$f(t) = \lim_{b\to\infty} \frac{1}{2\pi i} \int_{K_b} f^*(p)e^{pt}dp. \tag{3.51}$$

But on K_b we have

$$p = Re^{i\phi}(-\alpha \leqslant \phi \leqslant \alpha),$$

where

$$0 < \alpha = \arc\cos \frac{s}{R} < \frac{\pi}{2}.$$

Hence

$$\left| \int_{K_b} f^*(p) e^{pt} \mathrm{d}p \right| \leqslant \int_{-\alpha}^{\alpha} |f^*(Re^{i\phi})| \,.\, e^{Rt\cos\phi} \,.\, R\mathrm{d}\phi \leqslant$$

$$\leqslant \max_{-\alpha \leqslant \phi \leqslant \alpha} |f^*(Re^{i\phi})| R \int_{-\alpha}^{\alpha} e^{Rt\cos\phi} \mathrm{d}\phi. \tag{3.52}$$

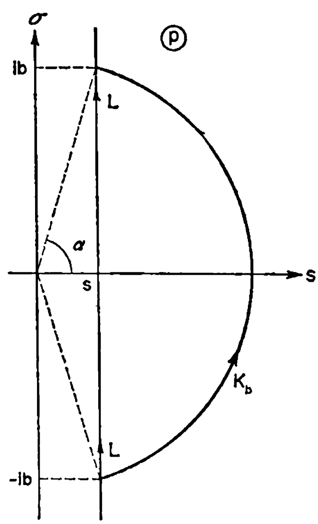


FIG. 17

Also, as for $0 \leqslant \phi \leqslant \pi/2$ the inequality $\cos\phi \geqslant 1 - 2\phi/\pi$, holds, taking into account that $t < 0$, we shall have

$$e^{Rt\cos\phi} \leqslant e^{Rt} e^{-(2R/\pi)\phi}.$$

Consequently,

$$R \int_{-\alpha}^{\alpha} e^{Rt\cos\phi} \mathrm{d}\phi = 2R \int_{0}^{\alpha} e^{Rt\cos\phi} \mathrm{d}\phi < 2Re^{Rt} \int_{0}^{\pi/2} e^{-(2Rt/\pi)\phi} \mathrm{d}\phi =$$

$$= 2Re^{Rt} \,.\, \frac{\pi}{2Rt}(1 - e^{-Rt}) = \frac{\pi}{-t}(1 - e^{Rt}) < \frac{\pi}{-t}$$

In addition to this,

$$\lim_{R\to\infty} \max_{-\alpha\leqslant\phi\leqslant\alpha} |f^*(Re^{i\phi})| = 0$$

by (3.50). Thus, the inequality (3.51) shows, that for $t < 0$

$$\lim_{R\to\infty} \int_{K_b} f^*(p)e^{pt}dp = 0,$$

and as when $b \to \infty$, $R \to \infty$ also, it follows by (3.51) that $f(t) = 0$

21. Expansion theorems and some applications

The inversion formula (3.40), proved in the preceding article under certain conditions, imposed on the Laplace transform $f^*(p)$, is in some cases not sufficiently convenient for the calculation of primary functions. However, if it is assumed, that $f^*(p)$ is representable in the form of a series

$$f^*(p) = \sum_{n=0}^{\infty} f_n^*(p),$$

which is convergent in a certain half-plane Re $p > s_0$, where both the function $f^*(p)$, and all the terms of the series $f_n^*(p)$ satisfy the conditions of theorem 7, then by (3.40) for $t > 0$

$$f(t) = \frac{1}{2\pi i}\int_L \sum_{n=0}^{\infty} f_n^*(p)e^{pt}dp = \sum_{n=0}^{\infty}\frac{1}{2\pi i}\int_L f_n^*(p)e^{pt}dp,$$

if the term by term integration of this series is permissible.

Let us suppose, that this is so. Then, if the primary functions for the $f_n^*(p)$ are known: $f_n^*(p) = \mathrm{L}f_n(t)$, then by the uniqueness theorem (theorem 3) we conclude, that

$$f_n(t) = \frac{1}{2\pi i}\int_L f_n^*(p)e^{pt}dp$$

and

$$f(t) = \sum_{n=0}^{\infty} f_n(t) \qquad (t > 0).$$

On this principle are based the following two theorems, called expansion theorems.

THEOREM 8. *If $f^*(p)$ is representable in the neighbourhood of the point at infinity in the form of the series*

$$f^*(p) = \sum_{n=0}^{\infty} c_n p^{-\gamma-nr-1}, \tag{3.53}$$

where $\gamma > -1$ and $r > 0$, which is convergent for $|p| > R$, and $p^{-\nu} = e^{-\nu \ln p}(-\pi < \operatorname{Im} \ln p \leqslant \pi)$ then for all $t > 0$

$$f(t) = \sum_{n=0}^{\infty} \frac{c_n}{\Gamma(\gamma+nr+1)} t^{\gamma+nr}.$$

In fact, the term by term integration of the series (3.53), multiplied by e^{pt}, along the path L with abscissa $s > R$ is permissible, as a power series can be integrated term by term in the domain of its convergence, and the path L is situated inside the domain $|p| > R$.

The conditions of theorem 7 are satisfied both for every term of the series (3.53), and for its sum $f^*(p)$. Condition 1 is obviously satisfied. Condition 3 is satisfied by virtue of the fact that

$$\left|\frac{f^*(p)}{p}\right| < \frac{A}{|p|^{\gamma+2}}$$

where A is some constant, and $\gamma+2 > 1$. As for condition 2, it is sufficient to verify it for every term of the series separately, as a power series converges uniformly in every interior part of its domain of convergence, and the sum of a uniformly convergent series, the terms of which are continuous functions is itself a continuous function. But

$$\int_{-\infty}^{\infty} \frac{1}{(s+i\sigma)^{\nu}} e^{i\sigma t} d\sigma$$

converges absolutely, if $\nu > 1$, and, consequently, is automatically

a continuous function of t. If however $0 < \nu < 1$, then, integrating by parts, we find that

$$\int_{-\infty}^{\infty} \frac{1}{(s+i\sigma)^{\nu}} e^{i\sigma t} d\sigma = \frac{1}{it} \frac{1}{(s+i\sigma)^{\nu}} e^{i\sigma t} \Big|_{\sigma=-\infty}^{\sigma=\infty} +$$

$$+ \frac{\nu}{t} \int_{-\infty}^{\infty} \frac{1}{(s+i\sigma)^{\nu+1}} e^{i\sigma t} d\sigma = \frac{\nu}{t} \int_{-\infty}^{\infty} \frac{1}{(s+i\sigma)^{\nu+1}} e^{i\sigma t} d\sigma,$$

where the last integral already converges absolutely.

Therefore, $f^*(p) = \mathbf{L}f(t)$ where,

$$f(t) = \sum_{n=0}^{\infty} c_n \cdot \frac{1}{2\pi i} \int_L p^{-\gamma-nr-1} e^{pt} dp,$$

and this formula is proved both for $t > 0$, and also for $t < 0$ (for which $f(t) = 0$), as $p^{\gamma-nr-1}$ satisfies the condition (3.50) (see remark 4 to theorem 7). But by formula (3.7) $p^{-\gamma-1}$ is the Laplace transform of the primary function $\eta(t)(t^{\nu}/\Gamma(\nu+1))$, so that

$$\frac{1}{2\pi i} \int_L p^{-\gamma-nr-1} e^{pt} dp = \eta(t) \frac{t^{\gamma+nr}}{\Gamma(\gamma+nr+1)}$$

and, consequently,

$$f(t) = \eta(t) \sum_{n=0}^{\infty} \frac{c_n}{\Gamma(\gamma+nr+1)} t^{\gamma+nr},$$

which is what it was required to prove.

Let us pass to some examples of the application of theorem 8.

Example 1. *A representation of* Cit, *which does not contain improper integrals.* We had (see (3.29)) the following formula:

$$\mathbf{L}\,\mathrm{Ci}\,t = -\frac{1}{2p} \ln(1+p^2).$$

As

$$-\frac{1}{2p}\ln(1+p^2) = -\frac{1}{2p}\left\{\ln p^2+\ln\left(1+\frac{1}{p^2}\right)\right\} =$$

$$= -\frac{1}{p}\ln p-\frac{1}{2}\sum_{n=1}^{\infty}(-1)^{n-1}\frac{p^{-2n-1}}{n},$$

and by (3.12)

$$-\frac{1}{p}\ln p = \mathbf{L}\{\ln t+C\},$$

where C is Euler's constant, it follows that

$$\mathbf{L}(\mathrm{Ci}\,t-\ln t-C) = \sum_{n=1}^{\infty}(-1)^n\frac{p^{-2n-1}}{2n}.$$

By theorem 8

$$\sum_{n=1}^{\infty}(-1)^n\frac{p^{-2n-1}}{2n} = \mathbf{L}\left\{\sum_{n=1}^{\infty}(-1)^n\frac{t^{2n}}{2n\,.\,(2n)!}\right\},$$

and

$$\sum_{n=1}^{\infty}(-1)^n\frac{t^{2n}}{2n\,.\,(2n)!} = \int_0^t\sum_{n=1}^{\infty}(-1)^n\frac{u^{2n-1}}{(2n)!}du = -\int_0^t\frac{1-\cos u}{u}du.$$

Thus,

$$\mathrm{Ci}\,t = \ln t+C-\int_0^t\frac{1-\cos u}{u}du.$$

Putting in this formula $t = 1$ and taking into account that

$$\mathrm{Ci}\,1 = -\int_1^{\infty}\frac{\cos u}{u}du,$$

we find, by the way, the following representation of Euler's constant:

$$C = \int_0^1 \frac{1-\cos u}{u} du - \int_1^\infty \frac{\cos u}{u} du.$$

Example 2. *A representation of* li $\exp(-t)$, *which does not contain improper integrals.* Formula (3.30) can be rewritten in the form

$$-\mathbf{L}f(t) = \mathbf{L}\mathrm{li}\, e^{-t} = -\frac{1}{p}\ln p - \frac{1}{p}\ln\left(1+\frac{1}{p}\right) =$$

$$= \mathbf{L}(\ln t + C) - \sum_{n=1}^{\infty}(-1)^{n-1}\frac{p^{-n-1}}{n}.$$

Consequently, by theorem 8

$$\mathbf{L}(-f(t) - \ln t - C) = \mathbf{L}(\mathrm{li}\, e^{-t}\ln t - C) =$$

$$= -\mathbf{L}\sum_{n=1}^{\infty}(-1)^{n-1}\frac{t^n}{n\,.\,n!} = -\mathbf{L}\int_0^t \sum_{n=1}^{\infty}(-1)^{n-1}\frac{u^{n-1}}{n!}du =$$

$$= -\mathbf{L}\int_0^t \frac{1-e^{-u}}{u}du.$$

Hence it follows, that

$$-f(t) = \mathrm{li}\, e^{-t} = \ln t + C - \int_0^t \frac{1-e^{-u}}{u}du.$$

Putting in this formula $t = 1$, we find that

$$C = \int_0^1 \frac{1-e^{-u}}{u}du - \int_1^\infty \frac{e^{-u}}{u}du,$$

as

$$f(1) = \int_1^\infty \frac{e^{-u}}{u}du.$$

Example 3. *The power series expansion of the function* $e t^2\Phi(t)$. By formula (3.32)

$$\mathbf{L}e^t\Phi(\sqrt{t}) = \frac{1}{\sqrt{(p)}(p-1)} = \frac{1}{p^{\frac{3}{2}}\left(1-\frac{1}{p}\right)} = \sum_{n=0}^{\infty} p^{-n-\frac{3}{2}},$$

and by theorem 8

$$\sum_{n=0}^{\infty} p^{-n-\frac{3}{2}} = \mathbf{L}\sum_{n=0}^{\infty}\frac{t^{n+\frac{1}{2}}}{\Gamma(n+\frac{3}{2})} = \mathbf{L}2\sqrt{\frac{t}{\pi}}\sum_{n=0}^{\infty}\frac{n!}{(2n+1)!}(4t)^n.$$

Consequently,

$$e^t\Phi(\sqrt{t}) = 2\sqrt{\frac{t}{\pi}}\sum_{n=0}^{\infty}\frac{n!}{(2n+1)!}(4t)^n,$$

or, replacing t by t^2,

$$e^{t^2}\Phi(t) = \frac{1}{\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{n!}{(2n+1)!}(2t)^{2n+1}\dagger).$$

THEOREM 9. *Let* $q_n (n = 1, 2, \dots)$ *be arbitrary complex numbers,* $\operatorname{Re} q_n < s_0$, m_n *be positive integers and*

$$P_n(p) = \sum_{k=1}^{m_n} A_k^{(n)}(p-q_n)^{k-1}$$

† This formula can be written as follows:

$$\Phi(t) = \frac{2}{\sqrt{\pi}}\int_0^t \exp(-u^2)\,du = \frac{1}{\sqrt{\pi}}\exp(-t^2)\sum_{n=0}^{\infty}\frac{n!}{(2n+1)!}(2t)^{2n+1}$$

or

$$\int_0^t \exp(-u^2)\,du = \tfrac{1}{2}\exp(-t^2)\sum_{n=0}^{\infty}\frac{n!}{(2n+1)!}(2t)^{2n+1},$$

which can, of course, be easily verified by differentiation.

be polynomials of degree not higher than $m_n - 1$. *Let us put*

$$f_n^*(p) = \frac{P_n(p)}{(p-q_n)^{m_n}}$$

and assume, that the series

$$f^*(p) = \sum_{n=1}^{\infty} f_n^*(p) = \sum_{n=1}^{\infty} \frac{P_n(p)}{(p-q_n)^{m_n}} \tag{3.54}$$

converges in the half-plane $\operatorname{Re} p > s_0$, *and* $f^*(p)$ *satisfies the conditions of theorem* 7. *Let us assume also, that the series* (3.54) *can after multiplication by* e^{pt} *be integrated term by term along the path* L: $\operatorname{Re} p = s > s_0$, *that is, that*

$$\frac{1}{2\pi i} \int_L f^*(p) e^{pt} dp = \sum_{n=1}^{\infty} \frac{1}{2\pi i} \int_L f_n^*(p) e^{pt} dp. \tag{3.55}$$

Then $f^*(p) = \mathbf{L} f(t)$ *where for* $t > 0$

$$f(t) = \sum_{n=1}^{\infty} f_n(t),$$

and

$$f_n(t) = \left\{ \sum_{k=1}^{m_n} \frac{A_k^{(n)}}{(m_n-k)!} t^{m_n-k} \right\} e^{q_n t}. \tag{3.56}$$

The statement of the theorem follows from equation (3.55), as its left hand side is equal to $f(t)$ (for $t > 0$), and each of the integrals on the right hand side can be represented in the form

$$\frac{1}{2\pi i} \int_L \sum_{k=1}^{m_n} \frac{A_k^{(n)}}{(p-q_n)^{m_n-k+1}} e^{pt} dp =$$

$$= \sum_{k=1}^{m_n} \frac{A_k^{(n)}}{(m_n-k)!} \cdot \frac{1}{2\pi i} \int_L \frac{(m_n-k)!}{(p-q_n)^{m_n-k+1}} e^{pt} dp.$$

But as the function $(p-q_n)^{-m_n+k-1}$ satisfies the conditions of theorem 7, it follows by (3.11) that

$$\frac{1}{2\pi i}\int\limits_L \frac{(m_n-k)!}{(p-q_n)^{m_n-k+1}}e^{pt}dp = t^{m_n-k}e^{q_n t},$$

and formula (3.56) is proved.

Example 4. *The primary function of a rational Laplace transform.* A particularly important case of theorem 9 is that, where $f^*(p)$ is the proper rational function

$$f^*(p) = \frac{H(p)}{Q(p)},$$

where, hence, the degree of the polynomial H is less than the degree of the polynomial Q. Then, if the q_n are the roots of $Q(p)$, and m_n are their multiplicities. the expansion (3.54) will hold, in which all the polynomials $P_n(p)$, beginning with a certain n, will be identically equal to zero:

$$f^*(p) = \frac{H(p)}{Q(p)} = \sum_{n=1}^{N} \frac{P_n(p)}{(p-q_n)^{m_n}}, \tag{3.54'}$$

where $m_1 + \ldots + m_N =$ the degree of $Q(p)$. In this case the infinite series (3.54), obviously converges to a finite sum, for which all the assumptions of theorem 9 are automatically satisfied, and we obtain the following formula:

$$\frac{H(p)}{Q(p)} = \mathbf{L}\sum_{n=1}^{N}\left\{\sum_{k=1}^{m_n}\frac{A_k^{(n)}}{(m_n-k)!}t^{m_n-k}\right\}e^{q_n t}. \tag{3.57}$$

The coefficients $A_k^{(n)}$ are determined by the rules for the expansion of rational functions into partial fractions well known from algebra. They can also be determined by the general formula, which is derived from the expansion (see (3.54'))

$$\frac{H(p)}{Q(p)} = \sum_{n=1}^{N}\sum_{k=1}^{m_n}\frac{A_k^{(n)}}{(p-q_n)^{m_n-k+1}} \tag{3.58}$$

in the following way. Let us choose the arbitrary $\nu = 1, \ldots, N$, and multiply the identity (3.58) by $(p-q_\nu)^{m_\nu}$. Then it can be written in the form

$$\frac{(p-q_\nu)^{m_\nu}}{Q(p)}H(p) = \sum_{k=1}^{m_\nu} A_k^{(\nu)}(p-q_\nu)^{k-1} +$$

$$+(p-q_\nu)^{m_\nu} \sum_{n=1}^{N}{}' \sum_{k=1}^{m_n} \frac{A_k^{(n)}}{(p-q_n)^{m_n-k+1}},$$

where the dash on the sum denotes, that in the summation with respect to n the value $n = \nu$ must be omitted (as the corresponding term has been written separately). Let us differentiate this identity $\kappa-1$ times with respect to p, where $\kappa = 1, \ldots, m_\nu$. Then we find, that

$$\frac{\mathrm{d}^{\kappa-1}}{\mathrm{d}p^{\kappa-1}} \left\{ \frac{(p-q_\nu)^{m_\nu}}{Q(p)}H(p) \right\} =$$

$$= \sum_{k=\kappa}^{m_\nu} A_k^{(n)}(k-1) \ldots (k-\kappa+1)(p-q_\nu)^{k-\kappa} +$$

$$+ \frac{\mathrm{d}^{\kappa-1}}{\mathrm{d}p^{\kappa-1}} \left\{ (p-q_\nu)^{m_\nu} \sum_{n=1}^{N}{}' \sum_{k=1}^{m_n} \frac{A_k^{(n)}}{(p-q_n)^{m_n-k+1}} \right\}.$$

Passing in this identity to the limit as $p \to q_\nu$, we find, that

$$\lim_{p\to q_\nu} \frac{\mathrm{d}^{\kappa-1}}{\mathrm{d}p^{\kappa-1}} \left\{ \frac{(p-q_\nu)^{m_\nu}}{Q(p)}H(p) \right\} = (\kappa-1)!A_k^{(\nu)},$$

and consequently,

$$A_k^{(n)} = \frac{1}{(k-1)!} \lim_{p\to q_n} \frac{\mathrm{d}^{k-1}}{\mathrm{d}p^{k-1}} \left\{ \frac{(p-q_n)^{m_n}}{Q(p)}H(p) \right\}. \tag{3.59}$$

If all the roots q_n of the polynomial $Q(p)$ are simple, that is every $m_n = 1$, then $P_n(p)$ reduces to a constant $A_1^{(n)}$, and the expansion (3.58) has the form

$$\frac{H(p)}{Q(p)} = \sum_{n=1}^{N} \frac{A_1^{(n)}}{p-q_n}.$$

In this formula the coefficients $A_1^{(n)}$ are found in a particularly simple way. Multiplying this identity by $p-\nu_\nu(\nu = 1, \ldots, N)$ and passing to the limit as $p \to q_\nu$, we find, that

$$\lim_{p\to q_\nu} \frac{p-q_\nu}{Q(p)} H(p) = A_1^{(\nu)},$$

and as $Q(q_\nu) = 0$, it follows that

$$\lim_{p\to q_\nu} \frac{Q(p)}{p-q_\nu} = \lim_{p\to q_\nu} \frac{Q(p)-Q(q_\nu)}{p-q_\nu} = Q'(q_\nu)$$

and consequently,

$$A_1^{(n)} = \frac{H(q_n)}{Q'(q_n)}$$

(let us note, that $Q'(q_n) \neq 0$, as q_n is a simple root). This formula can, of course, also be obtained from the general formula (3.59) for $k = 1$.

Thus, in the case of simple roots of the denominator

$$\frac{H(p)}{Q(p)} = \mathbf{L} \sum_{n=1}^{N} \frac{H(q_n)}{Q'(q_n)} e^{q_n t}. \tag{3.60}$$

Example 5. *The integration of ordinary linear homogeneous differential equations with constant coefficients.*† We have seen (see example 4, Art. 19), that the Laplace transforms of the solutions of such equations ((3.3.4) for $f(t) = 0$) are the proper rational functions:

$$y_0^*(p) = \frac{H(p)}{Q(p)}.$$

If by $y_0(t)$ is denoted the general solution of the homogeneous equation

$$y^{(\mu)} + a_1 y^{(\mu-1)} + \ldots + a_n y = 0,$$

† We shall not deal with non-homogeneous equations, as the Laplace transforms of their solutions (see (3.35), the term $y_H^*(p)$), as a rule (if the absolute term is not represented in the form of a linear combination of expressions of the form $t^n \exp(\lambda + i\omega)t)$), are not rational functions and require for their treatment other theorems, for which we cannot stop (for example, the shifting theorem; see A. I. Lur'ye, *The Operational Calculus*, Gostekhizdat, 1950, page 24).

then $\mathrm{L}y_0(t) = y_0^*(p)$, that is by (3.57)

$$y_0(t) = \sum_{n=1}^{N} M_n(t)e^{q_n t}, \tag{3.61}$$

where the q_n are the roots of the characteristic polynomial $Q(p)$, the m_n are their multiplicities, and the $M_n(t)$ are polynomials of degrees $m_n - 1$ with arbitrary coefficients. The fact, that formula (3.61) is actually the general solution of the homogeneous equation considered, follows from the fact, that the coefficients $A_k^{(n)}$, entering into the polynomials $M_n(t)$, are uniquely determined by formula (3.59) for any set of initial conditions, that is, for any $b_0, b_1, \ldots, b_{\mu-1}$, by which the polynomial $H(p)$ is constructed (see example 5, Art. 19). Let us note, that the number of arbitrary constants in (3.61) equals $m_1 + \ldots + m_N = \mu =$ the degree of $Q(p)$, that is the order of equation (3.24).

All these results are well known from the theory of linear differential equations. A certain advantage of the method given here in comparison with the classical method of integration of linear differential equations with constant coefficients consist in the fact that formula (3.61) already represents a solution, which satisfies the given initial conditions, while in Euler's theory the particular solution has to be obtained from the general solution by way of the solution of a linear system of algebraic equations for the arbitrary constants.

Let us consider a concrete example. Let it be required to find the solution of the linear equation

$$y^{(N)} + y = 0,$$

which satisfies the initial conditions: $y^{(n)}(0) = 0$, if $n \neq \nu$, $0 \leqslant n \leqslant N-1$, and $y^{(\nu)}(0) = 1$, $\nu = 0, 1, \ldots N-1$. As in the given case $\mathrm{L}y^{(N)}(t) = p^N \mathrm{L}y(t) - p^{N-\nu-1}$, it follows that for $\mathrm{L}y(t) = y^*(p)$ there is obtained the equation $(p^N + 1)y^*(p) = p^{N-\nu-1}$, that is,

$$y^*(q) = \frac{p^{N-\nu-1}}{p^N + 1}.$$

Consequently, here, $Q(p) = p^N + 1$, and $H(p) = p^{N-\nu-1}$. The polynomial $Q(p)$ has the simple roots

$$q_n = e^{[(2n-1)\pi i/N]} \qquad (n = 1, \ldots, N).$$

Thus, by (3.60)

$$y = \sum_{n=1}^{N} \frac{\exp[\{(N-\nu-1)/N\}(2n-1)\pi i]}{\exp[\{(N-1)/N\}(2n-1)\pi i]} \exp(e^{(2n-1)\pi i/N}t)$$

$$= \frac{1}{N}\sum_{n=1}^{N} \exp\left(-\frac{\nu}{N}(2n-1)\pi i\right)\exp\left(\cos\frac{2n-1}{N}\pi\right)t + i\left(\sin\frac{2n-1}{N}\pi\right)t. \tag{3.62}$$

This expression is, of course, real. In order to put it into real form, let us consider separately the cases of odd and even N.

First let $N = 2m$. Then let us split the sum (3.62) into two: with respect to n from 1 to m and from $m+1$ to $2m$; in the second of these sums let us introduce a new index of summation: $n' = 2m-n+1$, which consequently, runs through the values from 1 to m:

$$y = \frac{1}{2m}\sum_{n=1}^{m} \exp\left(-\frac{\nu}{2m}(2n-1)\pi i\right)\exp\left\{\left(\cos\frac{2n-1}{2m}\pi\right)t + i\left(\sin\frac{2n-1}{2m}\pi\right)t\right\}$$

$$+\frac{1}{2m}\sum_{n'=1}^{m} \exp\left(\frac{\nu}{2m}(2n'-1)\pi i\right)\exp\left\{\left(\cos\frac{2n'-1}{2m}\pi\right)t - i\left(\sin\frac{2n'-1}{2m}\pi\right)\right\}t,$$

or, denoting in the last sum the index of summation once more by n instead of by n' (as the value of the sum does not depend on the notation for the index of summation), we find, that

$$y = \frac{1}{2m}\sum_{n=1}^{m} \exp\left(\cos\frac{2n-1}{m}\pi\right)t \,.\, 2\mathrm{Re}\left[\exp\left\{i\left(\sin\frac{2n-1}{2m}\pi\right)t - \right.\right.$$

$$\left.\left.-\frac{\nu}{2m}(2n-1)\pi\right)\right\}\right] =$$

$$= \frac{1}{m}\sum_{n=1}^{m} \exp\left[\left(\cos\frac{2n-1}{2m}\pi\right)\right]t. \cos\left\{\left(\sin\frac{2n-1}{2m}\pi\right)t - \frac{\nu}{2m}(2n-1)\pi\right\}.$$

If however $N = 2m+1$, then in the sum (3.62) let us take out the term, which corresponds to the value $n = m+1$, and separate

the remaining sum into two: with respect to n from 1 to m and from $m+2$ to $2m+1$; in the second sum let us introduce a new index of summation: $n' = 2m-n+2$, which runs through the values from 1 to m:

$$y = \frac{1}{2m+1}e^{-\nu\pi i}e^{-t}+$$

$$+\frac{1}{2m+1}\sum_{n=1}^{m}\exp\left[-\frac{\nu}{2m+1}(2n-1)\pi i\right]$$

$$\times\exp\left(\cos\frac{2n-1}{2m+1}\pi\right)t+i\left(\sin\frac{2n-1}{2m+1}\pi\right)t+$$

$$+\frac{1}{2m+1}\sum_{n'=1}^{m}\exp\left[\frac{\nu}{2m+1}(2n'-1)\pi i\right]$$

$$\times\exp\left(\cos\frac{2n'-1}{2m+1}\pi\right)t-i\left(\sin\frac{2n'-1}{2m+1}\pi\right)t =$$

$$= (-1)^{\nu}\cdot\frac{1}{2m+1}e^{-t}+$$

$$+\frac{2}{2m+1}\sum_{n=1}^{m}\exp\left[\left(\cos\frac{2n-1}{2m+1}\pi\right)t\right]$$

$$\times\cos\left[\left(\sin\frac{2n-1}{2m+1}\pi\right)t-\frac{\nu}{2m+1}(2n-1)\pi\right],$$

which is also the final form of the solution in this case.

CHAPTER IV

CONTOUR INTEGRATION AND ASYMPTOTIC EXPANSIONS

In the present chapter the inversion formula of the Laplace transformation is applied first to the derivation of convergent expansions of primary functions in negative powers of the variable, and then to the derivation of what are known as asymptotic expansions.

The practical aspect of asymptotic expansions is considered in detail, and also the asymptotic expansions of some special functions.

The inversion formula of the Laplace transformation is applied to the derivation of the integral representation of Bessel's cylinder functions.

One of the important methods, which give asymptotic expansions of functions, representable in the form of definite parametric integrals, is the so-called saddle point method, which gives asymptotic expressions for such integrals for large values of the parameter. This method, also based on contour integration, is discussed at the end of the chapter and applied to the derivation of the asymptotic expansion of the gamma function.

22. The convergent expansion of the primary function in negative powers of the variable

Let us pass to transformations of the inversion formula (3.40), in which an important part is played by the deformation of the path of integration L together with Cauchy's theorem and the residue theorem,† that is by the method of contour integration. Here great importance will attach to the fact, that the Laplace transform $f^*(p)$ is in many cases a many-valued (algebraic or transcendental) function, the domain of regularity of any of its single-valued branches is the plane with cuts and with the exclusion of isolated points (poles or essential singularities).

Let us begin with a theorem, the proof of which contains some considerations which are typical of this kind of result.

† See F.C.V., Chap. V, Art. 48.

THEOREM 10. (THE RULE OF FRACTIONAL INDICES.) *Let $f^*(p)$ have the unique finite singular point $p = 0$, which is an algebraic or logarithmic branch point. Let us also assume, that $f^*(p)$ satisfies the conditions of theorem 7 (for $s_0 = 0$) and that, in addition to this,*

$$f^*(p) \to 0 \quad \text{as} \quad |p| \to \infty, \quad \operatorname{Re} p \leqslant s'_0 \ (s'_0 > 0). \tag{4.1}$$

Then, if

$$f^*(p) = p^\gamma \sum_{n=0}^{\infty} c_n p^{nr}, \tag{4.2}$$

where γ and r are real numbers ($r > 0$), where by p^α must be understood $\exp(\alpha \ln p)(-\pi < \operatorname{Im} \ln p \leqslant \pi)$, *and if the integral*

$$\int_0^\infty \sum_{n=0}^{\infty} |c_n||p|^{\gamma+nr}\, e^{-|p|t}\, \mathrm{d}|p|$$

converges for all $t > t_0 \geqslant 0$, then $f^*(p) = \mathrm{L}f(t)$, where for $t > t_0$

$$f(t) = \sum_{n=0}^{\infty} \frac{c_n}{\Gamma(-\gamma-nr)} t^{-\gamma-nr-1}. \tag{4.3}$$

Before passing to the proof of this theorem, let us make the following remark, explaining its title. If $\gamma+nr$ is a positive integer or zero, then $1/\Gamma(-\gamma-nr) = 0$,† and the corresponding integral power of t in the expansion (4.3) will be absent. Thus, in (4.3) there will remain only powers with fractional indices (and with any indices, greater than -1), so that this formula can be rewritten in the form

$$f(t) = \sum_{n=0}^{\infty}{}' \frac{c_n}{\Gamma(-\gamma-nr)} t^{-\gamma-nr-1}, \tag{4.3'}$$

where the dash on the sum in the given case denotes, that n only runs through those values, for which $\gamma+nr$ is not equal either to zero, or to a positive integer.

† See F.C.V., Chap. VII, Art. 80.

In order to prove the theorem let us note in the first place, that by (3.40) for $t > 0$ the required primary function is representable in the form

$$f(t) = \frac{1}{2\pi i}\int\limits_{L} f^*(p)\, e^{pt}\, \mathrm{d}p = \lim_{b\to\infty}\frac{1}{2\pi i}\int\limits_{L_b} f^*(p)\, e^{pt}\, \mathrm{d}p, \qquad (4.4)$$

where the path L runs in the right p-half plane at a distance s from the imaginary axis, where $0 < s < s_0$ and L_b is the segment $-b \leqslant \sigma \leqslant b$ of the path L. Also, $f^*(p)$ is regular in the p-plane cut along the negative real semi-axis. Let us now consider the contour

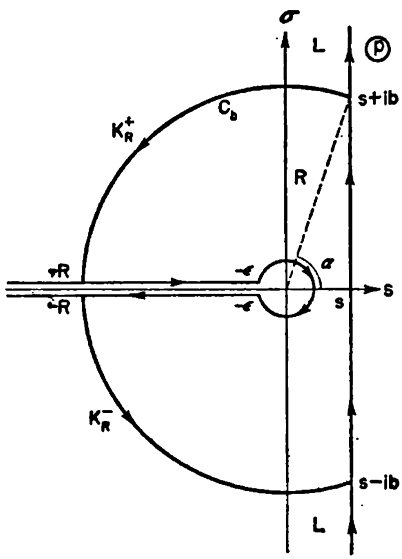


FIG. 18

C_b (Fig. 18), consisting of the segment L_b, of the arc $K_R{}^+$ of the circle $|p| = R = \sqrt{(s^2+b^2)}$, connecting the point $s+ib$ with the point $-R$ of the upper edge of the cut, of the path H_R, consisting of the segment $(-R, -\epsilon)$ of the upper edge of the cut, of the circle $|p| = \epsilon < s$, going round the origin, the segment $(-\epsilon, R)$ of the lower edge of the cut and, finally, the arc $K_R{}^-$ of the circle $|p| = R = \sqrt{(s^2+b^2)}$, connecting the point $-R$ of the lower edge of the cut with the point $s-ib$.

As, within and on the contour C_b, $f^*(p)e^{pt}$ is regular, it follows that

$$\frac{1}{2\pi i}\oint\limits_{C_b} f^*(p)\, e^{pt}\, \mathrm{d}p = 0$$

and, consequently, by (4.4)

$$\begin{aligned} f(t) = &-\lim_{b\to\infty}\frac{1}{2\pi i}\int\limits_{K_R+} f^*(p)\, e^{pt}\, \mathrm{d}p - \\ &-\lim_{b\to\infty}\frac{1}{2\pi i}\int\limits_{H_R} f^*(p)\, e^{pt}\, \mathrm{d}p - \\ &-\lim_{b\to\infty}\frac{1}{2\pi i}\int\limits_{K_R^-} f^*(p)\, e^{pt}\, \mathrm{d}p. \end{aligned} \tag{4.5}$$

We shall show, that the first and third terms in this formula are equal to zero. As on $K_R{}^+$ we have $p = Re^{i\phi}(\alpha \leqslant \phi \leqslant \pi)$, where $\cos\alpha = (s/R)(0 < \alpha < \pi/2)$, it follows that

$$\left|\int\limits_{K_R+} f^*(p)\, e^{pt}\, \mathrm{d}p\right| \leqslant R\int\limits_{\alpha}^{\pi} |f^*(Re^{i\phi})|e^{Rt\cos\phi}\, \mathrm{d}\phi = I_1 + I_2, \tag{4.6}$$

where

$$I_1 = R\int\limits_{\alpha}^{\pi/2} |f^*(Re^{i\phi})|e^{Rt\cos\phi}\, \mathrm{d}\phi,$$

$$I_2 = R\int\limits_{\pi/2}^{\pi} |f^*(Re^{i\phi})|e^{Rt\cos\phi}\, \mathrm{d}\phi,$$

So far as the first of these expressions is concerned, it is obvious that

$$\begin{aligned} I_1 &\leqslant R\max_{\alpha\leqslant\phi\leqslant\pi/2} |f^*(Re^{i\phi})|e^{Rt\cos\phi}\left(\frac{\pi}{2}-\alpha\right) \\ &= R\left(\frac{\pi}{2}-\alpha\right) e^{st}\max_{\alpha\leqslant\phi\leqslant\pi/2} |f^*(Re^{i\phi})|, \end{aligned}$$

as $R\cos\alpha = s$. But $R(\pi/2-\alpha) = R \arcsin \delta/R \to s$, and

$$\max_{\alpha\leqslant\phi\leqslant\pi/2} |f^*(Re^{i\phi})| \to 0$$

as $R \to \infty$, the latter in virtue of the fact, that for $\alpha \leqslant \phi \leqslant \pi/2$ and $p = R\,e^{i\phi}$ we have $\operatorname{Re} p = R\cos\phi \leqslant R\cos\alpha = s < s'_0$, and for such p the relation (4.1) is satisfied. Consequently, $I_1 \to 0$ as $R \to \infty$. Also,

$$I_2 \leqslant R \max_{\pi/2\leqslant\phi\leqslant\pi} |f^*(Re^{i\phi})| \int\limits_{\pi/2}^{\pi} e^{Rt\cos\phi}\,\mathrm{d}\phi;$$

but (putting $\phi = \pi/2+\psi$ and considering, that $t > 0$) it is easy to find the bound

$$R\int\limits_{\pi/2}^{\pi} e^{Rt\cos\phi}\,\mathrm{d}\phi = R\int\limits_{0}^{\pi/2} e^{-Rt\cos\phi}\,\mathrm{d}\psi \leqslant R\int\limits_{0}^{\pi/2} e^{-(2Rt/\pi)\phi}\,\mathrm{d}\psi$$

$$= \frac{\pi}{2t}(1-e^{-Rt}) < \frac{\pi}{2t}$$

and, hence, $I_2 \to 0$ as $R \to \infty$ in virtue of condition (4.1). By (4.6) we now in fact find that

$$\int\limits_{K_R+} f^*(p)e^{pt}\,\mathrm{d}p \to 0 \qquad \text{as} \qquad R \to \infty.$$

Similarly we find, that

$$\int\limits_{K_R} f^*(p)e^{pt}\,\mathrm{d}p \to 0 \qquad \text{as} \qquad R \to \infty.$$

As when $b \to \infty$ we also have $R \to \infty$, the relation (4.5), shows that

$$f(t) = \frac{1}{2\pi i}\int\limits_{H} f^*(t)e^{pt}\,\mathrm{d}p, \tag{4.7}$$

where H is the path, obtained from H_R by changing the orientation (connected with change of sign in front of the integral) and passing

to the limit as $R \to \infty$, that is the path, going from $-\infty$ along the lower edge of the cut to the point $-\epsilon$, then along the circumference $|p| = \epsilon$ to the point $-\epsilon$ of the upper edge of the cut and, finally, along the upper edge of the cut back to $-\infty$.

Having represented $f^*(p)$ in the form †

$$f^*(p) = \sum_{n=0}^{N} c_n p^{\gamma+nr} + r^*_N(p),$$

we can put formula (4.7) in the form

$$f(t) = \frac{1}{2\pi i} \int_H \sum_{n=0}^{N} c_n p^{\gamma+nr} e^{pt} \, \mathrm{d}p + \frac{1}{2\pi i} \int_H r^*_N(p) \, e^{pt} \, \mathrm{d}p, \tag{4.8}$$

where

$$\frac{1}{2\pi i} \int_H \sum_{n=0}^{N} c_n p^{\gamma+nr} e^{pt} \, \mathrm{d}p$$

$$= \sum_{n=0}^{N} c_n \cdot \frac{1}{2\pi i} \int_H p^{\gamma+nr} e^{pt} \, \mathrm{d}p$$

$$= \sum_{n=0}^{N} c_n t^{-\gamma-nr-1} \cdot \frac{1}{2\pi i} \int_{H'} \zeta^{\gamma+nr} e^{\zeta} \, \mathrm{d}\zeta,$$

where $\zeta = pt$ and H' is the path obtained from H by a magnification in the ratio $1:t$ with centre at the origin (see Fig. 19); the path H' differs from H only by the fact that it goes round the origin along a path of radius ϵt). But‡

$$\frac{1}{2\pi i} \int_{H'} \zeta^{\gamma+nr} e^{\zeta} \, \mathrm{d}\zeta = \frac{1}{\Gamma(-\gamma-nr)} \tag{4.9}$$

† Let us note, that in the given case this notation must not be understood as, that $r_N^*(p)$ is the Laplace transform of some primary function.

‡ See F.C.V., Chap. VIII, Art. 82. The path H' is there denoted by C_1.

and, hence,

$$\sum_{n=0}^{\infty} c_n \cdot \frac{1}{2\pi i} \int_H p^{\gamma+nr} e^{pt}\, \mathrm{d}p = \sum_{n=0}^{N} \frac{c_n}{\Gamma(-\gamma-nr)} t^{-\gamma-nr-1}. \quad (4.10)$$

Let us now show that

$$\lim_{N\to\infty} \int_H r^*_N(p)\, e^{pt}\, \mathrm{d}p = 0 \quad (4.11)$$

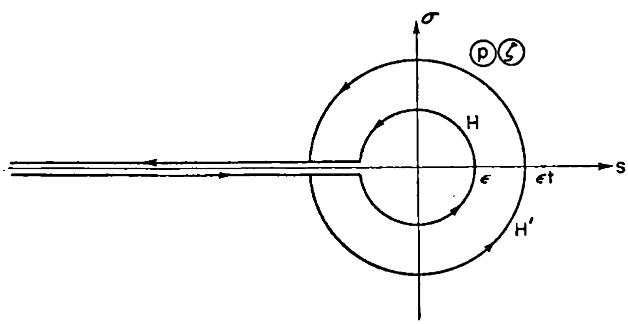


FIG. 19

for any value of $t > t_0$. In fact,

$$\left|\int_H r^*_N(p)\, e^{pt} \mathrm{d}p\right| \leqslant \left|\int_{H_R} r^*_N(p)\, e^{pt}\, \mathrm{d}p\right| + \left|\int_{H-H_R} r^*_N(p)\, e^{pt}\, \mathrm{d}p\right|, \quad (4.12)$$

where $H-H_R$ denotes the path consisting of the segment of the lower edge of the cut from $-\infty$ to $-R$ and of the segment of the upper edge of the cut from $-R$ to $-\infty$. But for any $t > t_0$

$$\left|\int_{H-H_R} r_N^*(p)\, e^{pt}\, \mathrm{d}p\right| \leqslant$$

$$\leqslant 2 \int_R^{\infty} \sum_{n=N+1}^{\infty} |c_n| |p|^{\gamma+nr} e^{-|p|t}\, \mathrm{d}|p| \leqslant$$

$$\leqslant 2 \int_R^{\infty} \sum_{n=0}^{\infty} |c_n| |p|^{\gamma+nr} e^{-|p|t}\, \mathrm{d}|p|,$$

as, in virtue of the convergence by the conditions of the theorem of the last integral, R can be chosen so great, that for all N

$$\left| \int_{H-H_R} r^*_N(p)\, e^{pt}\, \mathrm{d}p \right| < \tfrac{1}{2}\eta,$$

where $\eta > 0$ is a number as small as desired. Having chosen R in this way, let us note, that the series (4.2) converges in the circle $|p| \leqslant R$ (as this series converges for all $|p| < \infty$). Hence N can now be chosen so great, that for all $|p| \leqslant R$

$$|r^*_N(p)| = \left| \sum_{n=N+1}^{\infty} c_n p^{\gamma+nr} \right| < \frac{1}{2A}\eta,$$

where

$$A = \int_{H_R} |e^{pt}||\mathrm{d}p|.$$

Then

$$\left| \int_{H_R} r^*_N(p)\, e^{pt}\, \mathrm{d}p \right| < \frac{1}{2A}\eta \int_{H_R} |e^{pt}||\mathrm{d}p| = \tfrac{1}{2}\eta,$$

and by (4.12) for all sufficiently great N and $t > t_0$

$$\left| \int_{H} r^*_N(p)\, e^{pt}\, \mathrm{d}p \right| < \eta,$$

which also indicates the correctness of equation (4.11).

Passing now in formula (4.8) to the limit as $N \to \infty$ and taking account of the relation (4.10), we finally find that for $t > t_0$

$$f(t) = \sum_{n=0}^{\infty} \frac{c_n}{\Gamma(-\gamma-nr)} t^{-\gamma-nr-1},$$

that is formula (4.3).

Example 1. *The primary function of the Laplace transform* $1/\sqrt{(p)}e^{-\lambda\sqrt{(p)}}$, where $\lambda > 0$ *and* $\mathrm{Re}\sqrt{p} \geqslant 0$.

For $f^*(p) = 1/\sqrt{(p)}e^{-\lambda\sqrt{(p)}}$ the conditions of theorem 10 are satisfied,

$$f^*(p) = \sum_{n=0}^{\infty} \frac{(-\lambda)^n}{n!} p^{(n-1)/2},$$

with $t_0 = 0$, as

$$\sum_{n=0}^{\infty} \frac{\lambda^n}{n!} |p|^{(n-1)/2} = \frac{1}{\sqrt{|p|}} e^{\lambda\sqrt{|p|}}$$

and

$$\int_0^{\infty} \frac{1}{\sqrt{|p|}} e^{\lambda\sqrt{|p|}} \cdot e^{-|p|t}\, \mathrm{d}|p|,$$

obviously converges for all $t > 0$. Consequently in the corresponding formula (4.3′) for the primary function, n must only run through even values $n = 2m$. Therefore, for $t > 0$

$$f(t) = \sum_{m=0}^{\infty} \frac{(-\lambda)^{2m} t^{-(2m-1)/2-1}}{(2m)!\Gamma[-(2m-1)/2]} = \sum_{m=0}^{\infty} \frac{\lambda^{2m} t^{-m-\frac{1}{2}}}{(2m)!\Gamma(-m+\frac{1}{2})}.$$

But†

$$\Gamma(-m+\tfrac{1}{2}) = \frac{\Gamma(-m+\frac{3}{2})}{-m+\frac{1}{2}} = \frac{\Gamma(-m+\frac{5}{2})}{(-m+\frac{1}{2})(-m+\frac{3}{2})} = \ldots =$$

$$= \frac{\Gamma(\frac{1}{2})}{(-m+\frac{1}{2})(-m+\frac{3}{2}) \ldots (-\frac{1}{2})} =$$

$$= \frac{2^m\sqrt{\pi}}{(-1)^m 1\,.\,3 \ldots (2m-1)} = (-1)^m \cdot \frac{2^{2m} m!}{(2m)!} \sqrt{\pi}$$

and, hence,

$$f(t) = \frac{1}{\sqrt{(\pi t)}} \sum_{m=0}^{\infty} (-1)^m \frac{(\lambda^2/4t)^m}{m!} = \frac{1}{\sqrt{(\pi t)}} e^{-\lambda 2/4t}.$$

† See F.C.V., Chap. VII, Art. 80.

Thus,

$$\frac{1}{\sqrt{p}}e^{-\lambda\sqrt{p}} = \mathrm{L}\frac{1}{\sqrt{(\pi t)}}e^{-\lambda^2/4t},$$

as $\lambda > 0$ and $\mathrm{Re}\sqrt{p} > 0$.

This result, however, indicates that

$$\frac{1}{\sqrt{p}}e^{-\lambda\sqrt{p}} = \int_0^\infty \frac{1}{\sqrt{\pi t}}e^{-\lambda^2/4t}e^{-pt}\mathrm{d}t =$$

$$= \frac{1}{\sqrt{\pi}}\int_0^\infty e^{-(\lambda^2/4t)-pt}\frac{\mathrm{d}t}{\sqrt{t}} = \frac{2}{\sqrt{\pi}}\int_0^\infty e^{-(\lambda^2/4u^2)-pu^2}\mathrm{d}u$$

where $u = \sqrt{t}$. But $-\lambda^2/4u^2-pu^2 = -(\lambda/2u-\sqrt{(p)}u)^2-\lambda\sqrt{p}$. Consequently,

$$\frac{2}{\sqrt{\pi}}\int_0^\infty e^{-(\lambda^2/4u^2)-pu^2}\mathrm{d}u = \frac{2}{\sqrt{\pi}}\int_0^\infty e^{-[(\lambda/^2u)-\sqrt{(p)}u]^2}e^{-\lambda\sqrt{(p)}}\mathrm{d}u,$$

and we find, that

$$\frac{2}{\sqrt{\pi}}\int_0^\infty e^{-[(\lambda/2u)-\sqrt{(p)}u]^2}\mathrm{d}u = \frac{1}{\sqrt{p}}.$$

Considering p to be real and positive and putting $\sqrt{(p)}u = v$, $\lambda/2\sqrt{p} = \alpha$, we find, that

$$\frac{2}{\sqrt{\pi}}\int_0^\infty e^{-[(\alpha/v)-v]^2}\mathrm{d}u = 1,$$

that is, that the integral on the left hand side of this equation does not depend on α.

Example 2. *The primary function of the Laplace transform* $(1/p)e^{-\lambda\sqrt{p}}$, *where* $\lambda > 0$ *and* $\mathrm{Re}\sqrt{p} \geqslant 0$.

The function

$$f^*(p) = \frac{1}{p}e^{-\lambda\sqrt{p}}$$

satisfies the conditions of theorem 10, with

$$f^*(p) = \sum_{n=0}^{\infty} \frac{(-\lambda)^n}{n!} p^{(n/2)-1}.$$

Consequently, in the corresponding formula (4.3′) for the primary function the index n must assume the value zero and odd integral values: $n = 0$ and $n = 2m+1$. Thus, for $t > 0$

$$f(t) = 1 + \sum_{m=0}^{\infty} \frac{(-\lambda)^{2m+1} t^{-m-\frac{1}{2}}}{(2m+1)!\Gamma(-m+\frac{1}{2})} =$$

$$= 1 - \sum_{m=0}^{\infty} \frac{\lambda^{2m+1} t^{-m-\frac{1}{2}}}{(2m+1)!\Gamma(-m+\frac{1}{2})}.$$

But, as we saw above in example 1,

$$\Gamma(-m+\tfrac{1}{2}) = (-1)^m \frac{2^{2m} m!}{(2m)!} \sqrt{\pi},$$

and hence

$$f(t) = 1 - \frac{2}{\sqrt{\pi}} \sum_{m=0}^{\infty} (-1)^m \frac{(\lambda/2\sqrt{t})^{2m+1}}{(2m+1)m!}.$$

Putting $\lambda/2\sqrt{(r)} = v$, we shall have

$$\sum_{m=0}^{\infty} (-1)^m \frac{(\lambda/2\sqrt{t})^{2m+1}}{(2m+1)m!} = \sum_{m=0}^{\infty} (-1)^m \frac{1}{m!} \int_0^v u^{2m} du = \int_0^v e^{-u^2} du,$$

and hence, finally (see example 3, Art .19)

$$f(t) = 1 - \frac{2}{\sqrt{\pi}} \int_0^{\lambda/2\sqrt{t}} e^{-u^2} du = 1 - \Phi(\lambda/2\sqrt{t}).$$

Thus,

$$\frac{1}{p} e^{-\lambda\sqrt{p}} = \mathbf{L}\left[1 - \Phi\left(\frac{\lambda}{2\sqrt{t}}\right)\right].$$

23. Asymptotic expansions and their connexion with contour integration

Equation (4.3) of the preceding article is a convergent series for all $t > t_0$. Consequently if we put

$$f(t) = \sum_{n=0}^{N} \frac{c_n}{\Gamma(-\gamma-nr)} t^{-\gamma-nr-1} + r_N(t), \tag{4.13}$$

then

$$\lim_{N\to\infty} r_N(t) = 0 \tag{4.14}$$

for any $t > t_0$. In what follows we shall see, that considerations, entirely similar to those which were used in the proof of theorem 10, lead with somewhat different assumptions about $f^*(p)$ to an equation of the form (4.13), in which, however, the remainder term $r_N(r)$ does not satisfy condition (4.14) not even for a single value of $t > 0$, but instead of this, the relation

$$\lim_{t\to\infty} t^{\gamma+Nr+1} r_N(t) = 0 \tag{4.15}$$

holds for any $N \geqslant 0$, that is, as $t \to \infty$ the remainder term is of a higher order of smallness, than the last term of the partial sum. In such a case the series

$$\sum_{n=0}^{\infty} \frac{c_n}{\Gamma(-\gamma-nr)} t^{-\gamma-nr-1}$$

will not, of course, converge for a single value of $t > 0$. Nevertheless the formula (4.13) with the property (4.15) can be fully used for the approximate calculation of $f(t)$ for large values of t and, thus, has serious practical value. In fact, for fixed $t > 0$

$$\left| f(t) - \sum_{n=0}^{N} \frac{c_n}{\Gamma(-\gamma-nr)} t^{-\gamma-nr-1} \right| \leqslant |r_N(t)|.$$

Let us assume, that there exists a number N_t, such that $|r_{N_r}(t)| \leqslant$ $\leqslant r_N(t)|$ for all $N \geqslant 0$. Then $|r_{N_t}(t)| = \epsilon(t)$ is, obviously the least

error, with which, for a given value of t, it is possible for $f(t)$ to be approximated to by a sum of the form

$$\sum_{n=0}^{N} \frac{c_n}{\Gamma(-\gamma-nr)} t^{-\gamma-nr-1},$$

this error being attained for $N = N_t$. From the condition (4.15) it follows, that $\epsilon(t) \to 0$ as $t \to \infty$, that is, that this least error tends to zero as $t \to \infty$.

If $f(t)$ is representable in the form (4.13), *and condition* (4.15) *is satisfied, it is usual to write this as follows:*

$$f(t) \sim \sum_{n=0}^{\infty} \frac{c_n}{\Gamma(-\gamma-nr)} t^{-\gamma-nr-1}, \tag{4.16}$$

and this last series (generally speaking, divergent for every $t > 0$) is called the asymptotic expansion of $f(t)$.

If we introduce the notation: $t^r = \tau$, $t^{\gamma+1}f(t) = \phi(\tau)$,

$$\frac{c_n}{\Gamma(-\gamma-nr)} = a_n,$$

the asymptotic expansion (4.16) assumes the form

$$\phi(\tau) \sim \sum_{n=0}^{\infty} a_n \tau^{-n},$$

where

$$\lim_{\tau\to\infty} \tau^N \left\{ \phi(\tau) - \sum_{n=0}^{N} a_n \tau^{-n} \right\} = 0.$$

Hence for $N = 0$ it follows that

$$\lim_{\tau\to\infty} \{\phi(\tau) - a_0\} = 0,$$

that is,

$$a_0 = \lim_{\tau\to\infty} \phi(\tau), \tag{4.17}$$

for $N = 1$ that

$$\lim_{\tau\to\infty} \tau \left\{ \phi(\tau) - a_0 - \frac{a_1}{\tau} \right\} = 0,$$

that is,

$$a_1 = \lim_{\tau\to\infty} \tau \{\phi(\tau) - a_0\},$$

and generally, for $n = 1, 2, \ldots$ that

$$a_n = \lim_{\tau\to\infty} \tau^n \left\{ \phi(\tau) - a_0 - \frac{a_1}{\tau} - \ldots - \frac{a_{n-1}}{\tau^{n-1}} \right\}. \qquad (4.18)$$

Thus, every function, defined for $t > 0$, can have only one asymptotic expansion, as the coefficients of this expansion are uniquely determined (if this expansion exists in general) by formulas (4.17) and (4.18). From these formulas it follows, in particular, that for $\delta > 0$ the function $C\,e^{-\delta t}$, where C is an arbitrary constant, has an asymptotic expansion, all the coefficients of which are equal to zero.†

A typical example of an asymptotic expansion is the following. Let us consider the function

$$\phi(\tau) = \tau^{\nu-1} e^{\tau} \int_{\tau}^{\infty} e^{-u} u^{-\nu} \mathrm{d}u.$$

Integrating by parts, we find, that

$$\int_{\tau}^{\infty} e^{-u} u^{-\nu} \mathrm{d}u = e^{-\tau}\tau^{-\nu} - \nu \int_{\tau}^{\infty} e^{-u} u^{-\nu-1} \mathrm{d}u = \ldots =$$

$$= e^{-\tau}\tau^{-\nu} - \nu e^{-\tau}\tau^{-\nu-1} + \ldots +$$
$$+ (-1)^{N-1}\nu(\nu+1)\ldots(\nu+N-2) e^{-\tau}\tau^{-\nu-N+1} +$$
$$+ (-1)^{N}\nu(\nu+1)\ldots(\nu+N-1) \int_{\tau}^{\infty} e^{-u} u^{-\nu-N} \mathrm{d}u,$$

† This, however, indicates that the asymptotic expansion does not determine the function uniquely, as the functions

$$\phi(\tau) + C\exp(-\delta t) \qquad \text{and} \qquad \phi(\tau)$$

have one and the same asymptotic expansion.

and consequently,

$$\phi(\tau) = \tau^{-1} - \frac{\Gamma(\nu+1)}{\Gamma(\nu)}\tau^{-2} + \dots$$
$$\dots + (-1)^{N-1}\frac{\Gamma(\nu+N-1)}{\Gamma(\nu)}\tau^{-N} + r_N(\tau),$$

where

$$r_N(\tau) = (-1)^N \frac{\Gamma(\nu+N)}{\Gamma(\nu)}\tau^{\nu-1}e^{\tau}\int_{\tau}^{\infty} e^{-u}u^{-\nu-N}du.$$

As

$$\int_{\tau}^{\infty} e^{-u}u^{-\nu-N}du < \tau^{-\nu-N}\int_{\tau}^{\infty} e^{-u}du = \tau^{-\nu-N}e^{-\tau},$$

it follows that

$$|r_N(\tau)| < \frac{\Gamma(\nu+N)}{\Gamma(\nu)}\tau^{-N-1}$$

and $\tau^N r_N(\tau) \to 0$ as $\tau \to \infty$, and consequently,

$$\phi(\tau) = \tau^{\nu-1}e^{\tau}\int_{\tau}^{\infty} e^{-u}u^{-\nu}du \sim \sum_{n=0}^{\infty}\frac{\Gamma(\nu+n)}{\Gamma(\nu)}\tau^{-n-1}.$$

Let us investigate this asymptotic expansion for concrete numerical data. Let, for example, $\nu = 1$, $\tau = 10$. Then by the preceding

$$|r_N(10)| = N!\int_{10}^{\infty} e^{10-u}u^{-N-1}du < N!\int_{10}^{\infty} u^{-N-1}du = \frac{(N-1)!}{10^N}.$$

The least value of $(N-1)!/10^N$ for integral $N \geqslant 1$ will, as is not difficult to verify, be $9!/10^{11} < 0{\cdot}00004$, which is attained for $N = 10$. Consequently,

$$\left| e^{10}\int_{10}^{\infty}\frac{e^{-u}}{u}du - \left\{\frac{1}{10} - \frac{1}{10^2} + \frac{2!}{10^3} - \frac{3!}{10^4} + \frac{4!}{10^5} - \dots\right.\right.$$
$$\left.\left.\dots + \frac{8!}{10^0} - \frac{9!}{10^{10}}\right\}\right| < 0{\cdot}00004,$$

and the difference, the absolute value of which stands on the left hand side of this inequality, has the sign $(-1)^{10}$, that is, it is positive. Thus (see example 2, Art. 19),

$$\int_{10}^{\infty} \frac{e^{-u}}{u} du = (0{\cdot}09154 \ldots + \eta)e^{-10},$$

where $0 < \eta < 0{\cdot}00004$.

Let us now turn to the Laplace transform $f^*(p)$ and its primary function $f(t)$ and consider, under what conditions and how, instead of the convergent expansion (4.3), we obtain for $f(t)$ the asymptotic expansion (4.16).

THEOREM 11. *Let $f^*(p)$ have besides the singular point $p = 0$, which is an algebraic or logarithmic branch point, also a finite number of singular points $p_1, p_2, \ldots p_k (k \geqslant 1)$, which are not branch points*† *and do not lie on the real positive semi-axis of the p-plane. Let us also assume, that $f^*(p)$ satisfies the conditions of theorem 7 (for*

$$s_0 = \max\{\mathrm{Re}p_1, \ldots, \mathrm{Re}p_k\})$$

and that, in addition to this, also

$$f^*(p) \to 0 \quad \text{as } |p| \to \infty, \quad \mathrm{Re}p \leqslant s'_0 \quad (s'_0 > s_0). \tag{4.19}$$

Then if

$$f^*(p) = p^{\gamma} \sum_{n=0}^{\infty} c_n p^{nr}, \tag{4.20}$$

where $r > 0$, and by p^{α} must be understood $e^{\alpha \ln p} (-\pi < \mathrm{Im} \ln p \leqslant \pi)$, then $f^*(p) = \mathrm{L}f(t)$, and

$$f(t) - \sum_{\kappa=1}^{k} \mathrm{res}\, f^*(p_\kappa) e^{p_\chi t} \sim \sum_{n=0}^{\infty} \frac{c_n}{\Gamma(-\gamma - nr)} t^{-\gamma - nr - 1}. \tag{4.21}$$

Before passing to the proof of this theorem, let us note, that its assumptions differ from the assumptions of theorem 10 basically only by the fact that $f^*(p)$ has the singular points $p_1, \ldots, p_k (k \geqslant 1)$

† That is in the neighbourhood of each of the points $p_\chi (1 \leqslant \chi \leqslant k)$, $f^*(p)$ must be single valued.

and that as a result of this the expansion (4.20) converges only in the circle $|p| < \rho$, where $\rho = \min\{|p_1|, \ldots |p_k|\}$.

Let us consider the contour C_b, defined in the preceding article (see Fig. 18), and let us assume that $s > s_0$, $0 < \epsilon < \rho$ and $R = \sqrt{(s^2+b^2)} > \max\{|p_1|, \ldots, |p_k|\}$. Applying now to the contour C_b the fundamental theorem on residues, we find that

$$\frac{1}{2\pi i}\oint_{C_b} f^*(p)e^{pt}\mathrm{d}p = \sum_{\varkappa=1}^{k} \operatorname{res} f^*(p_\varkappa)e^{p_\varkappa t}.$$

As we saw above (see the proof of theorem 10, Art. 22), under the given conditions

$$\lim_{R\to\infty}\int_{K_R^+} f^*(p)e^{pt}\mathrm{d}p = \lim_{R\to\infty}\int_{K_R^-} f^*(p)e^{pt}\mathrm{d}p = 0,$$

so that

$$f(t) = \lim_{b\to\infty}\left\{\frac{1}{2\pi i}\oint_{C_b} f^*(p)e^{pt}\mathrm{d}p - \frac{1}{2\pi i}\int_{H_R} f^*(p)e^{pt}\mathrm{d}p\right\} =$$

$$= \sum_{\varkappa=1}^{k} \operatorname{res} f^*(p_\varkappa)e^{p_\varkappa t} + \frac{1}{2\pi i}\int_{H} f^*(p)e^{pt}\mathrm{d}p, \tag{4.22}$$

where H_R and H are the paths defined in the preceding article (see pages 205 and 208).

However in distinction from the continuation of the proof of theorem 10, the substitution in the integral along the path H in formula (4.22) of the expansion (4.20), is impossible in the given case, as this expansion converges only for $|p| < \rho$. In view of this, it is useful to proceed in the following way.

Let us introduce the new notation†

$$r^*_N(p) = f^*(p) - \sum_{n=0}^{N} c_n p^{\gamma+nr}. \tag{4.23}$$

† See the remark on page 136.

Then

$$\frac{1}{2\pi i}\int_H f^*(p)e^{pt}\mathrm{d}p =$$

$$= \sum_{n=0}^{N} c_n \cdot \frac{1}{2\pi i}\int_H p^{\gamma+nr}e^{pt}\mathrm{d}p + \frac{1}{2\pi i}\int_H r_N^*(p)e^{pt}\mathrm{d}p.$$

But for $t > 0$ (H' is the path represented on Fig. 19, page 188)

$$\frac{1}{2\pi i}\int_H p^{\gamma+nr}e^{pt}\mathrm{d}p = \frac{1}{t^{\gamma+nr+1}}\cdot\frac{1}{2\pi i}\int_{H'}\zeta^{\gamma+nr}e^{\zeta}\mathrm{d}\zeta = \frac{t^{-\gamma-nr-1}}{\Gamma(-\gamma-nr)}$$

by (4.9), and consequently,

$$\frac{1}{2\pi i}\int_H f^*(p)e^{pt}\mathrm{d}p = \sum_{n=0}^{N}\frac{c_n}{\Gamma(-\gamma-nr)}t^{-\gamma-nr-1} + r_N(t), \tag{4.24}$$

where†

$$r_N(t) = \frac{1}{2\pi i}\int_H r_N^*(p)e^{pt}\mathrm{d}p. \tag{4.25}$$

Let us find a bound for the last integral. Let H consist of the segments of the edges of the cut: $\arg p = -\pi(-\infty < p < -\epsilon)$ and $\arg p = \pi(-\epsilon > p > -\infty)$, which we will denote by Γ_ϵ^- and Γ_ϵ^+, respectively, and the circle $|p| = \epsilon(-\pi < \arg p < \pi)$, which we will denote by Γ_ϵ. As on Γ_ϵ^- and Γ_ϵ^+ there are no singular points of the functions $f^*(p)$ and $f^*(p) \to 0$ as $p \to -\infty$ along the real axis, then for fixed N the ratio

$$\frac{r_N^*(p)}{p^{\gamma+Nr}} = \frac{f^*(p)}{p^{\gamma+Nr}} - \sum_{n=0}^{N} c_n p^{-(N-n)r}$$

must be bounded in modulus on Γ_ϵ^- and Γ_ϵ^+:

$$\left|\frac{r_N^*(p)}{p^{\gamma+Nr}}\right| \leqslant A, \tag{4.26}$$

† As already noted, $r_N(p)$ need not be the primary function of $r_N^*(p)$.

where A is some constant, depending of course on ϵ and N. Then

$$\left|\frac{1}{2\pi i}\int_{\Gamma_\epsilon^-} r_N^*(p)e^{pt}\mathrm{d}p\right| \leqslant$$

$$\leqslant \frac{A}{2\pi}\int_{-\infty}^{-\epsilon} |p|^{\gamma+Nr}e^{pt}\mathrm{d}p = t^{-\gamma-Nr-1}\cdot\frac{A}{2\pi}\int_{\epsilon t}^{\infty} \xi^{\gamma+Nr}e^{-\xi}\mathrm{d}\xi. \tag{4.27}$$

A similar bound holds also for the corresponding integral along Γ_ϵ^+:

$$\left|\frac{1}{2\pi i}\int_{\Gamma_\epsilon^+} r_N^*(p)e^{pt}\mathrm{d}p\right| \leqslant t^{-\gamma-Nr-1}\cdot\frac{A}{2\pi}\int_{\epsilon t}^{\infty} \xi^{\gamma+Nr}e^{-\xi}\mathrm{d}\xi. \tag{4.28}$$

Consequently, it remains to evaluate the integral

$$\frac{1}{2\pi i}\int_{\Gamma_\epsilon} r_N^*(p)e^{pt}\mathrm{d}p.$$

As the circumference Γ_ϵ lies within the circle of convergence $|p| < \rho$ of the series (4.20), on Γ_ϵ (see (4.23))

$$r_N^*(p) = \sum_{n=N+1}^{\infty} c_n p^{\gamma+nr} = p^{\gamma+(N+1)r}\phi_N(p),$$

where

$$\phi_N^*(p) = \sum_{m=0}^{\infty} c_{N+1+m}p^{mr}.$$

It is not difficult to see, that $\phi_N^*(p)$ is bounded inside and on the circle Γ_ϵ:

$$|\phi_N^*(p)| \leqslant B, \tag{4.29}$$

where, of course, the constant B depends on ϵ and N. In fact, if

$$M = \max_{|p|=\rho'}\left|\frac{f^*(p)}{p^\gamma}\right| \qquad (\rho' < \rho),$$

then by Cauchy's inequalities for the coefficients†

$$|c_n| \leqslant \frac{M}{\rho'^{nr}}.$$

Let $0 < \epsilon < \rho' < \rho$. Then on Γ_ϵ

$$|\phi_N(p)| \leqslant \sum_{m=0}^{\infty} |c_{N+1+m}|\epsilon^{m\rho} \leqslant$$

$$\leqslant \frac{M}{\rho'^{(N+1)r}} \sum_{m=0}^{\infty} \left(\frac{\epsilon}{\rho'}\right)^{mr} = \frac{M}{\rho'^{(N+1)r}} \frac{1}{1-(\epsilon/\rho')^r}$$

and, of course, this bound must also hold inside Γ'_ϵ.

Let us now take into consideration the path h_ϵ (Fig. 20), which connects the beginning and end points of the path Γ_ϵ: $-\epsilon e^{-\pi i}$ and $-\epsilon e^{\pi i}$, and consists of the segment $\arg p = -\pi(-\epsilon < p < -\epsilon')$ of the lower edge of the cut, the segment $\arg p = \pi(-\epsilon' > p > -\epsilon)$ of the upper edge of the cut and the circle $|p| = \epsilon' < \epsilon(-\pi < < \arg p < \pi)$. In virtue of the regularity of the function in the domain, bounded by the lines Γ_ϵ and h_ϵ, we shall, by Cauchy's theorem have the equation

$$\frac{1}{2\pi i}\int_{\Gamma_\epsilon} r_N^*(p)e^{pt}dp = \frac{1}{2\pi i}\int_{k_\epsilon} r_N^*(p)e^{pt}dp. \tag{4.30}$$

In the last integral it is possible to pass to the limit as $\epsilon' \to 0$, when the integral along the circle $|p| = \epsilon'$ will have zero as its limit, if N is so great, that $\gamma+(N+1)r > -1$ (in what follows we shall consider only such values of N). In fact, on this circle $p = \epsilon' e^{i\phi}(-\pi < \phi < \pi)$,

$$|r_N^*(p)| = \epsilon'^{\gamma+(N+1)r}|\phi_N^*(\epsilon' e^{i\phi})$$

and $|e^{pt}| = e^{\epsilon' t\cos\phi}$, so that the absolute value of the integrand on this circle does not exceed

$$\epsilon'^{\gamma+(N+1)r}e^{\epsilon' t} \max_{-\pi\leqslant\phi\leqslant\pi} |\phi_N^*(\epsilon' e^{i\phi})|,$$

where

$$\max_{-\pi\leqslant\phi\leqslant\pi} |\phi_N^*(\epsilon' e^{i\phi})| \to |c_{N+1}|$$

† See F.C.V., Chap. VI, formula 31.

as $\epsilon' \to 0$. As the length of this circumference is equal to $2\pi\epsilon'$, the absolute value of the integral we are interested in does not exceed

$$\epsilon'^{\gamma+(N+1)r+1}e^{\epsilon' t} \max_{-\pi\leqslant\phi\leqslant\pi} |\phi_N^*(\epsilon' e^{i\phi})|,$$

which has zero as its limit as $\epsilon' \to 0$, if $\gamma+(N+1)r > -1$.

Thus, we have to find a bound for the integral

$$\frac{1}{2\pi i}\int_{h_\epsilon} r_N^*(p)e^{pt}\mathrm{d}p = \frac{1}{2\pi i}\int_{h_\epsilon} \phi_N^*(p)p^{\gamma+(N+1)r}e^{pt}\mathrm{d}p,$$

where h_ϵ is the path, which consists of the segments of the edges of

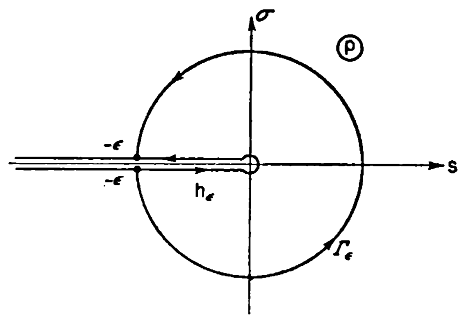


FIG. 20

the cut: $\arg p = -\pi(-\epsilon < p < 0)$ and $\arg p = \pi(0 > p > -\epsilon)$. By (4.29)

$$\left|\frac{1}{2\pi i}\int_{h_\epsilon} \phi_N^*(p)p^{\gamma+(N+1)r}e^{pt}\mathrm{d}p\right| < \frac{B}{2\pi}\int_{h_\epsilon} |p|^{\gamma+(N+1)r}e^{pt}|\mathrm{d}p| =$$

$$= \frac{B}{\pi}\int_{-\epsilon}^{0} (-s)^{\gamma+(N+1)r}e^{st}\mathrm{d}s =$$

$$= t^{-\gamma-(N+1)r-1}\cdot\frac{B}{\pi}\int_{0}^{\epsilon t} \xi^{\gamma+(N+1)r}e^{-\xi}\mathrm{d}\xi,$$

and as

$$\int_0^{\epsilon t} \xi^{\gamma+(N+1)r}e^{-\xi}\mathrm{d}\xi < \int_0^{\infty} \xi^{\gamma+(N+1)r}e^{-\xi}\mathrm{d}\xi = \Gamma\{\gamma+(N+1)r+1\},$$

taking account also of equation (4.30), we find, that

$$\left|\frac{1}{2\pi i}\int_{\Gamma_\epsilon} r_N{}^*(p)e^{pt}\mathrm{d}p\right| < \frac{B}{\pi}\Gamma\{\gamma+(N+1)r+1\}t^{-\gamma-(N+1)r-1}. \quad (4.31)$$

By virtue of (4.25) the bounds (4.27), (4.28) and (4.31) show, that

$$|r_N(t)| < t^{-\gamma-Nr-1}\left\{\frac{A}{\pi}\int_{\xi t}^{\infty}\xi^{\gamma+Nr}e^{-\xi}\mathrm{d}\xi + \right.$$

$$\left. + \frac{B}{\pi}\frac{\Gamma\{\gamma+(N+1)r+1\}}{t^r}\right\}. \quad (4.32)$$

Hence it follows, that

$$\lim_{t\to\infty} t^{\gamma+Nr+1}r_N(t) = 0,$$

and this, if (4.24) is considered, signifies, that

$$\frac{1}{2\pi i}\int_H f^*(p)e^{pt}\mathrm{d}p \sim \sum_{n=0}^{\infty}\frac{c_n}{\Gamma(-\gamma-nr)}t^{-\gamma-nr-1}.$$

Thus, by (4.22) we finally obtain that

$$f(t) - \sum_{\kappa=1}^{m}\operatorname{res} f^*(p_\kappa)e^{p_\kappa t} \sim \sum_{n=0}^{\infty}\frac{c_n}{\Gamma(-\gamma-nr)}t^{-\gamma-nr-1},$$

and the theorem is proved.

It is necessary to note that asymptotic expansions acquire practical value only in combination with an estimate of the remainder term of the type (4.32), as the relation (4.15) is not sufficient for purposes of calculation.

In the following examples a detailed analysis of the evaluation of the remainder terms of asymptotic expansions and their application to concrete calculations is given.

Example 1. *The asymptotic expansion of the complementary probability integral.* By the complementary probability integral we

shall understand the function (see example 3, Art. 19)

$$1-\Phi(t) = 1-\frac{2}{\sqrt{\pi}}\int\limits_0^t e^{-u^2}du = \frac{2}{\sqrt{\pi}}\int\limits_t^\infty e^{-u^2}du.$$

By (3.32)

$$\mathrm{L}e^t\Phi(\sqrt{t}) = f^*(p) = \frac{1}{\sqrt{(p)(p-1)}} = -p^{-\frac{1}{2}}\sum_{n=0}^{\infty} p^n.$$

The conditions of theorem 11 are satisfied. Consequently, by (4.21)

$$e^t\Phi(\sqrt{t})-e^t \sim -\sum_{n=0}^{\infty}\frac{t^{-n-\frac{1}{2}}}{\Gamma(-n+\frac{1}{2})}$$

or

$$e^t\{1-\Phi(\sqrt{t})\} \sim \sum_{n=0}^{\infty}\frac{t^{-n-\frac{1}{2}}}{\Gamma(-n+\frac{1}{2})}. \tag{4.33}$$

As $\Gamma(-n+\frac{1}{2}) = (-1)^n[(2^{2n}n!)/(2n)!]\sqrt{\pi}$ (see example 1, Art. 22), then replacing in (4.33) t by t^2, we obtain the asymptotic expansion

$$e^{t^2}\{1-\Phi(t)\} \sim \frac{2}{\sqrt{\pi}}\sum_{n=0}^{\infty}(-1)^n\frac{(2n)!}{n!}\left(\frac{2}{t}\right)^{2n+1}. \tag{4.34}$$

In order to evaluate the precision with which this asymptotic expansion can be used to evaluate the values of the function, which stands on the left hand side of it, for sufficiently large values of t, we shall make use of formula (4.32), from which in the given case ($\gamma = -\frac{1}{2}$ and $r = 1$)

$$|r_N(t)| < t^{-N-\frac{1}{2}}\left\{\frac{A}{\pi}\int\limits_{\epsilon t}^{\infty}\xi^{N-\frac{1}{2}}e^{-\xi}d\xi+\frac{B}{\pi}\frac{\Gamma(N+\frac{3}{2})}{t}\right\},$$

where $0 < \epsilon < 1$. The constant A must be chosen such that it

satisfies equation (4.26). As

$$r_N^*(p) = \frac{1}{\sqrt{(p)}(p-1)} + p^{-\frac{1}{2}} \sum_{n=0}^{N} p^n =$$

$$= \frac{1}{\sqrt{(p)}(p-1)} + \frac{p^{N+1}-1}{\sqrt{(p)}(p-1)} = \frac{p^{N+\frac{1}{2}}}{p-1},$$

this means, that we have to put $A = 1$, or

$$\left| \frac{r_N^*(p)}{p^{\gamma+Nr}} \right| = \left| \frac{p}{p-1} \right| < 1$$

on the real negative p-semi-axis. With regard to the constant B, for $|p| \leqslant \epsilon$ the inequality (4.29) must be satisfied. In the given case $\phi_N^*(p) = 1/(p-1)$, and obviously, as B we can take $1/(1-\epsilon)$. Thus

$$|r_N(t)| < \frac{t^{-N-\frac{1}{2}}}{\pi} \left\{ \int_{\epsilon t}^{\infty} \xi^{N-\frac{1}{2}} e^{-\xi} \mathrm{d}\xi + \frac{\Gamma(N+\frac{3}{2})}{t(1-\epsilon)} \right\}.$$

In this bound the value of ϵ remains arbitrary (but, of course, lying between 0 and 1); as ϵ is increased (without changing the values of t and N) the second expression in the curly brackets decreases, and the first increases. The optimum value of ϵ depends, of course, on t and N. It is laborious to calculate it (it is a root of a transcendental equation) and not worth while in practice. We can proceed in the following way: in view of the difficulty of a satisfactory bound for the first expression let us replace its lower limit of integration by zero (by which it is increased); then only the second expression will depend on ϵ, and we can pass to the limit as $\epsilon \to \infty$. As a result we obtain the bound

$$|r_N(t)| < \frac{t^{-N-\frac{1}{2}}}{\pi} \left\{ \int_{0}^{\infty} \xi^{N-\frac{1}{2}} e^{-\xi} \mathrm{d}\xi + \frac{\Gamma(N+\frac{3}{2})}{t} \right\} =$$

$$= \frac{1}{\pi} \left\{ \frac{\Gamma(N+\frac{1}{2})}{t^{N+\frac{1}{2}}} + \frac{\Gamma(N+\frac{3}{2})}{t^{N+\frac{3}{2}}} \right\},$$

as

$$\int_0^\infty \xi^{N-\frac{1}{2}}e^{-\xi}d\xi = \Gamma(N+\tfrac{1}{2}).$$

Also,

$$\Gamma(N+\tfrac{3}{2}) = (N+\tfrac{1}{2})(N-\tfrac{1}{2})\dots\tfrac{1}{2}\Gamma(\tfrac{1}{2}) =$$
$$= \frac{1\,.\,3\dots(2N+1)}{2^{N+1}}\sqrt{\pi} = \frac{(2N+2)!}{(N+1)!2^{2N+2}}\sqrt{\pi}$$

and

$$\Gamma(N+\tfrac{1}{2}) = \frac{\Gamma(N+\frac{3}{2})}{N+\frac{1}{2}}.$$

Consequently,

$$|r_N(t)| < \frac{2}{\sqrt{\pi}}\frac{(2N+2)!}{(N+1)!}\left(\frac{1}{2\sqrt{t}}\right)^{2N+3}\left\{\frac{2t}{2N+1}+1\right\}.$$

The remainder term in the expansion (4.34) is equal to $-r_N(t^2)$, as in passing from the original asymptotic expansion to the expansion (4.33) we changed all the signs into the opposite ones, and in passing from the expansion (4.33) to the expansion (4.34) we replaced t by t^2. Thus, the relation (4.34) is equivalent to the following:

$$e^{t^2}\{1-\Phi(t)\} = \frac{2}{\sqrt{\pi}}\sum_{n=0}^{N}(-1)^n\frac{(2n)!}{n!}\left(\frac{1}{2t}\right)^{2n+1} - r_N(t^2), \tag{4.35}$$

where

$$|r_N(t^2)| < \frac{2}{\sqrt{\pi}}\frac{(2N+2)!}{(N+1)!}\left(\frac{1}{2t}\right)^{2N+3}\left\{\frac{2t^2}{2N+1}+1\right\}. \tag{4.36}$$

In view of the great practical importance of the evaluation of remainder terms in asymptotic expansions (as also, by the way, in convergent series) we shall also consider in this example the possibility of the direct estimation of $r_N(t)$, starting not from the inequality (4.32) (which already contains a number of crude approximations), but from the representation (4.25). This is possible in those concrete cases, where $r_N{}^*(p)$ is a comparatively simple function. As a rule, there is obtained in this way an estimate of the remainder term, a great deal better, than the bound (4.32).

In the example considered

$$r_N{}^*(p) = \frac{p^{N+\frac{1}{2}}}{p-1}$$

and, hence

$$r_N(t) = \frac{1}{2\pi i}\int\limits_H \frac{p^{N+\frac{1}{2}}}{p-1} e^{pt}\mathrm{d}p.$$

In this integral we can pass to the limit as $\epsilon \to 0$, as the corresponding integral along Γ_ϵ (see page 199) then tends to zero. Then the path H reduces to

$\Gamma_0{}^-$: $\qquad \arg p = -\pi(\infty < p < 0)$

and

$\Gamma_0{}^+$: $\qquad \arg p = \pi(0 > p > -\infty)$.

On $\Gamma_0{}^-$ we have $p = |p|\ e^{-i\pi}$, on $\Gamma_0{}^+$ we have $p = |p|\ e^{i\pi}$ and, consequently,

$$r_N(t) = -\frac{1}{2\pi i}\int\limits_\infty^0 \frac{|p|^{N+\frac{1}{2}}e^{-i(N+\frac{1}{2})\pi}}{-|p|-1} e^{-|p|t}\mathrm{d}|p| -$$

$$-\frac{1}{2\pi i}\int\limits_0^\infty \frac{|p|^{N+\frac{1}{2}}e^{i(N+\frac{1}{2})\pi}}{-|p|-1} e^{-|p|t}\mathrm{d}|p| =$$

$$= \{e^{i(N+\frac{1}{2})\pi} - e^{-i(N+\frac{1}{2})\pi}\} \cdot \frac{1}{2\pi i}\int\limits_0^\infty \frac{|p|^{N+\frac{1}{2}}}{|p|+1} e^{-|p|t}\mathrm{d}|p|.$$

But

$$e^{i(N+\frac{1}{2})\pi} - e^{-i(N+\frac{1}{2})\pi} = 2i\sin(N+\tfrac{1}{2})\pi = 2i(-1)^N,$$

so that, putting $|p|\ t = \xi$, we shall have the equation

$$r_N(t) = (-1)^N \frac{t^{-N-\frac{1}{2}}}{\pi}\int\limits_0^\infty \frac{\xi^{N+\frac{1}{2}}}{\xi+t} e^{-\xi}\mathrm{d}\xi, \tag{4.37}$$

which, in the first place, gives the sign of the remainder term $r_N(t)$ and, in the second place, the following simple bound for it:

$$|r_N(t)| < \frac{t^{-N-\frac{3}{2}}}{\pi}\int_0^\infty \xi^{N+\frac{1}{2}}e^{-\xi}d\xi = \frac{1}{\pi}\frac{\Gamma(N+\frac{3}{2})}{t^{N+\frac{3}{2}}} =$$

$$= \frac{2}{\sqrt{\pi}}\frac{(2N+2)!}{(N+1)!}\left(\frac{1}{2\sqrt{t}}\right)^{2N+3}.$$

Thus,

$$|r_N(t^2)| < \frac{2}{\sqrt{\pi}}\frac{(2N+2)!}{(N+1)!}\left(\frac{1}{2t}\right)^{2N+3} \tag{4.36'}$$

This inequality shows that the *error made by replacing the function* $e^{t^2}\{1-\Phi(t)\}$ *by any section of its asymptotic expansion* (4.34), *does not exceed in absolute value the absolute value of the first neglected term of the expansion.*

Let us now evaluate with the help of the inequality (4.36′) the least possible error in the approximation to the function $e^{t^2}\{1-\Phi(t)\}$ by a section of the asymptotic series (4.34) for a given fixed value of t. By (4.35) for this it is necessary to find the least value of $|r_N(t^2)|$, when N runs through negative integral values. Let us put

$$A_N = \frac{(2N+2)!}{(N+1)!}\left(\frac{1}{2t}\right)^{2N+3}.$$

Then $(A_N/A_{N-1}) = (2N+1)/2t^2$ and, consequently, $A_N \leqslant A_{N-1}$ for $N+\frac{1}{2} \leqslant t^2$ and $A_N > A_{N-1}$ for $N+\frac{1}{2} > t^2$, that is, the least A_N has a suffix, which satisfies the inequality $N+\frac{1}{2} \leqslant t^2 < N+\frac{3}{2}$, that is, it is equal to the greatest integer, which does not exceed $t^2-\frac{1}{2}$ (it is assumed, that t is so great, that $t^2-\frac{1}{2} > 0$). Let us denote this number by N_t. Then by (4.36′)

$$|r_{N_t}(t^2)| < \frac{2}{\sqrt{\pi}}\frac{(2N_t+2)!}{(N_t+1)!}\left(\frac{1}{2t}\right)^{2N_t+3}. \tag{4.38}$$

As the absolute value of the general term of the asymptotic expansion (4.34) is proportional to A_{n-1}, the preceding discussion shows that *for the given fixed value of t the absolute values of the terms of the expansion* (4.34) *first decrease, and then, beginning with the*

term with suffix N_t+1, *again increase, and that the best approximation is achieved, if the asymptotic expansion is broken off at the term, which precedes the least one: the error made by this* (as is shown by the bound (4.38)) *will be less in absolute value than the absolute value of the next term of the expansion which, for the given value of t is the least one.*

The proposition, printed in italics above, holds for the majority of asymptotic expansions used in practice.

Let us note, that the corresponding bound derived from the inequality (4.36) gives an error approximately twice as great.

However, even the bound (4.36′) can be improved. In fact, by formula (4.37)

$$|r_N(t^2)| = \frac{t^{-2N-1}}{\pi}\int_0^\infty \frac{\xi^{N+\frac{1}{2}}}{\xi+t^2}e^{-\xi}\mathrm{d}\xi.$$

Let us write this equation in the following form:

$$|r_N(t^2)| =$$

$$= \frac{t^{-2N-3}}{\pi}\int_0^\infty \xi^{N+\frac{1}{2}}e^{-\xi}\mathrm{d}\xi\,.\,\frac{\displaystyle\int_0^\infty \frac{t^2}{\xi+t^2}\xi^{N+\frac{1}{2}}e^{-\xi}\mathrm{d}\xi}{\displaystyle\int_0^\infty \xi^{N+\frac{1}{2}}e^{-\xi}\mathrm{d}\xi} =$$

$$= \delta_N\frac{2}{\sqrt{\pi}}\frac{(2N+2)!}{(N+1)!}\left(\frac{1}{2t}\right)^{2N+3},$$

where

$$\delta_N = \frac{\displaystyle\int_0^\infty \frac{t^2}{\xi+t^2}\xi^{N+\frac{1}{2}}e^{-\xi}\mathrm{d}\xi}{\displaystyle\int_0^\infty \xi^{N+\frac{1}{2}}e^{-\xi}\mathrm{d}\xi}.$$

In these integrals let us put $\xi = (N+\frac{1}{2})x$; multiplying them by

$e^{N+\frac{1}{2}}$ and also putting $t^2 = \mu(N+\frac{1}{2})$, we find that

$$\delta_N = \frac{\displaystyle\int\limits_0^\infty \frac{\mu}{x+\mu}(xe^{1-x})^{N+\frac{1}{2}}\mathrm{d}x}{\displaystyle\int\limits_0^\infty (xe^{1-x})^{N+\frac{1}{2}}\mathrm{d}x}.$$

It will be shown below (see example 1, Art. 25), that

$$\lim_{N\to\infty} \sqrt{N} \int\limits_0^\infty (xe^{1-x})^{N+\frac{1}{2}}\mathrm{d}x = \sqrt{(2\pi)}$$

and

$$\lim_{N\to\infty} \sqrt{N} \int\limits_0^\infty \frac{\mu}{x+\mu}(xe^{1-x})^{N+\frac{1}{2}}\mathrm{d}x = \frac{\mu}{1+\mu}\sqrt{(2\pi)}.$$

Thus, $\delta_N \to \mu/(1+\mu)$ as $N \to \infty$,† and consequently, the value of δ_N for large N differs little from $\mu/(1+\mu)$. As for $N = N_t$, μ is only a little greater than unity, it follows from this that δ_{N_t} (for sufficiently great values of N, that is, of t) will be approximately equal to $\frac{1}{2}$. This shows, that, at least for large t the bound (4.36′) can be improved almost by a factor of two, that is, *that the error in the partial sum of the asymptotic series, which is broken off at the term with suffix N_t, will lie within limits close to half the absolute value of the first neglected (the least for the given fixed value of t) term of the expansion.*

Let us note, finally, that a value of $\delta_N < 1$ makes it possible significantly to improve the resulting approximation to the function. In fact, let

$$S_N = \sum_{n=0}^{N} (-1)^n u_n,$$

†It can be shown, that $\delta_N < \mu/(1+\mu)$ for sufficiently large N.

where

$$u_n = \frac{2}{\sqrt{\pi}} \frac{(2n)!}{n!} \left(\frac{1}{2t}\right)^{2n+1}.$$

As by (4.37) r_N is positive for even, and negative for odd, N and

$$e^{t^2}\{1-\Phi(t)\} = S_N - r_N,$$

where $|r_N| < \delta_N u_{N+1}$, it is obvious that, for example, for even N_t the value of $e^{t^2}(1-\Phi(t))$ lies between

$$S_{N_t} - r_{N_t} > S_{N_t} - \delta_{N_t} u_{N_t+1}$$

and S_{N_t}, and at the same time also between $S_{N_t+1} = S_{N_t} - u_{N_t+1}$ and

$$S_{N_t+1} + r_{N_t+1} < S_{N_t+1} + \delta_{N_t+1} u_{N_t+2}.$$

These two inequalities significantly compress the interval of possible

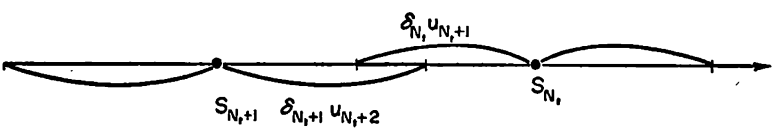


FIG. 21

values of the function (Fig. 21). It is easily seen that its length is equal to

$$\delta_{N_t} u_{N_t+1} + \delta_{N_t+1} u_{N_t+2} - (S_{N_t} - S_{N_t+1})$$
$$= \delta_{N_t} u_{N_t+1} + \delta_{N_t+1} u_{N_t+2} - u_{N_t+1}$$
$$= \delta_{N_t+1} u_{N_t+2} - (1-\delta_{N_t}) u_{N_t+1}.$$

If we consider that δ_{N_t} and δ_{N_t+1} differ little from $\frac{1}{2}$, the length of this interval can be considered as approximately equal to $\frac{1}{2}(u_{N_t+2} - u_{N_t+1})$, where u_{N_t+1} is for the given value of t the least term of the asymptotic expansion. A similar degree of precision will hold also for odd N_t.

The application of this remark gives, as a rule, exceptionally precise results. We shall not go in greater detail into the basis of this precision, as for this it would be necessary to consider also the asymptotic behaviour of δ_N.

Let us consider some numerical values. Let us put, for example, $t \doteq 2$. Then $N_t = 3$ and

$$e^4\{1-\Phi(2)\} = \frac{2}{\sqrt{\pi}}\left\{\frac{1}{4}-\frac{2!}{1!}\left(\frac{1}{4}\right)^3+\frac{4!}{2!}\left(\frac{1}{4}\right)^5-\frac{6!}{3!}\left(\frac{1}{4}\right)^7\right\}-r_3(4),$$

where $r_3(4)$ is negative and $|r_3(4)| < (2/\sqrt{\pi})(8!/4!)(\frac{1}{4})^9$. Thus,

$$e^4\{1-\Phi(2)\} = \frac{2}{\sqrt{\pi}}(0{\cdot}25-0{\cdot}03125+0{\cdot}011718\ldots-0{\cdot}007324\ldots)+$$

$$+|r_3(4)| = \frac{2}{\sqrt{\pi}}\cdot 0{\cdot}223144\ldots+|r_3(4)| = 0{\cdot}25180\ldots+|r_3(4)|,$$

where

$$|r_3(4)| < \frac{2}{\sqrt{\pi}}\cdot 0{\cdot}006408\ldots = 0{\cdot}00723\ldots$$

Hence,

$$0{\cdot}25180 < e^4\{1-\Phi(2)\} < 0{\cdot}25904,$$

that is

$$1-0{\cdot}25904\, e^{-4} < \Phi(2) < 1-0{\cdot}25180\, e^{-4},$$

or

$$0{\cdot}99525 < \Phi(2) < 0{\cdot}99538.$$

This inequality ensures three significant figures in $\Phi(2)$. In fact $\Phi(2) = 0{\cdot}99532\ldots$

The refinement given above enables us to find five significant figures of $\Phi(2)$. In fact, in the given case $S_{N_t} = 0{\cdot}25180$, , . . r_{N_t} is negative, and $|r_{N_t}| < 0{\cdot}00723\ldots$ Consequently, on the one hand, if we put $\delta_{Nt} = \frac{1}{2}$

$$e^4\{1-\Phi(2)\} = 0{\cdot}25180\ldots+\tfrac{1}{2}\cdot 0{\cdot}00723\ldots < 0{\cdot}25542.$$

On the other hand,

$$S_{N_t+1} = 0{\cdot}25180\ldots+0{\cdot}00723\ldots > 0{\cdot}25903,$$

r_{N_t+1} is positive and

$$r_{N_t+1} < \frac{2}{\sqrt{\pi}}\frac{10!}{5!}\left(\frac{1}{4}\right)^{11} < 0{\cdot}00814.$$

Thus, putting $\delta_{N_t+1} = \frac{1}{2}$, we shall have, that

$$e^4\{1-\Phi(2)\} > 0{\cdot}25903 - 0{\cdot}00407 = 0{\cdot}25496,$$

that is,

$$1 - 0{\cdot}25542\, e^{-4} < \Phi(2) < 1 - 0{\cdot}25496\, e^{-4},$$

or

$$0{\cdot}995321 < \Phi(2) < 0{\cdot}995329.$$

Let us note, that the calculation of $\Phi(2)$ could also be carried out with the help of the expansion of the function $e^{t^2}\Phi(t)$ in a series of positive powers of t which is convergent for all t (see example 3, Art. 21):

$$\Phi(2) = e^{-4} \cdot \frac{1}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{n!}{(2n+1)!} 4^{2n+1}$$

$$= e^{-4} \cdot \frac{1}{\sqrt{\pi}} \left(4 + \frac{4^3}{2 \cdot 3} + \frac{4^5}{3 \cdot 4 \cdot 5} + \ldots \right).$$

However this series converges so slowly, that, for example, in order to calculate $\Phi(2)$ to three significant figures it would be necessary to take up to 15 terms of this series. For greater values of t this series becomes practically useless, while the asymptotic expansion gives a result with the prescribed degree of precision which is the more precise (that is, with the smaller number of terms), the greater is t.

Example 2. *The asymptotic expansion of Fresnel's integrals.* In the same way as the asymptotic expansion of the complementary probability integral was analysed above, it is possible to obtain asymptotic expansions of the functions

$$\cos t \cdot C(\sqrt{t}) + \sin t \cdot S(\sqrt{t})$$

and

$$\sin t \cdot C(\sqrt{t}) - \cos t \cdot S(\sqrt{t}),$$

the Laplace transforms of which are, respectively,

$$\frac{\sqrt{p}}{p^2+1} \quad \text{and} \quad \frac{1}{\sqrt{p}(p^2+1)}$$

(see example 4, Art. 19). This function obviously satisfies the conditions of theorem 11.

As

$$\frac{\sqrt{p}}{p^2+1} = p^{\frac{1}{2}} \sum_{n=0}^{\infty} (-1)^n p^{2n}$$

and the function $[\sqrt{(p)}/(p^2+1)]e^{p^t}$ has two poles $p = \pm i$ with residues, respectively, equal to†

$$\frac{\sqrt{i}}{2i} e^{it} = \frac{1}{2i} e^{i[(\pi/4)+t]} \quad \text{and} \quad \frac{\sqrt{-i}}{-2i} e^{-it} = -\frac{1}{2i} e^{-i[(\pi/4)+t]},$$

by (4.21)

$$\cos t \cdot C(\sqrt{t}) + \sin t \cdot S(\sqrt{t}) - \frac{1}{2i}\{e^{i[(\pi/4)+t]} - e^{-i[(\pi/4)+t]}\} \sim$$

$$\sim \frac{2}{\sqrt{\pi}} \sum_{n=0}^{\infty} (-1)^n \frac{t^{-2n-3/2}}{\Gamma(-2n-\frac{1}{2})},$$

or, in view of the fact, that

$$\Gamma(-2n-\tfrac{1}{2}) = \frac{\Gamma(-2n+\frac{1}{2})}{-2n-\frac{1}{2}}$$

$$= (-1)^{2n} \frac{2^{4n}(2n)!}{(4n)!} \frac{\sqrt{\pi}}{-2n-\frac{1}{2}} = -\frac{2^{4n+1}(2n)!}{(4n+1)!}\sqrt{\pi},$$

we shall have the asymptotic expansion

$$\sin\left(t+\frac{\pi}{4}\right) - \cos t \cdot C(\sqrt{t}) - \sin t \cdot S(\sqrt{t}) \sim$$

$$\sim \frac{2}{\sqrt{\pi}} \sum_{n=0}^{\infty} (-1)^n \frac{(4n+1)!}{(2n)!} \left(\frac{1}{2\sqrt{t}}\right)^{4n+3}. \quad (4.39a)$$

Similarly

$$\frac{1}{\sqrt{p}(p^2+1)} = p^{-\frac{1}{2}} \sum_{n=0}^{\infty} (-1)^n p^{2n},$$

---

† As indicated above, by $\sqrt{p}$ must be understood that value of it, for which $Re\sqrt{(p)} > 0$. Hence we must put $\sqrt{(i)} = \exp(i(\pi/4))$ and so on.

the residues of the function $1/\sqrt{(p)}(p^2+1)\, e^{p^t}$ at the poles $p = \pm i$ are equal, respectively, to

$$\frac{1}{2i\sqrt{i}}e^{it} = \frac{1}{2i}e^{i[-(\pi/4)+t]}$$

and

$$\frac{1}{-2i\sqrt{-i}}\,e^{-t} = -\frac{1}{2i}\,e^{i[(\pi/4)-t]}$$

and, consequently,

$$\sin t \cdot C(\sqrt{t}) - \cos t \cdot S(\sqrt{t}) - \frac{1}{2i}\{e^{i[t-(\pi/4)]} - e^{-i[t-(\pi/4)]}\} \sim$$

$$\sim \sum_{n=0}^{\infty}(-1)^n \frac{t^{-2n-\frac{1}{2}}}{\Gamma(-2n+\frac{1}{2})},$$

or

$$\sin t \cdot C(\sqrt{t}) - \cos t \cdot S(\sqrt{t}) - \sin\left(t-\frac{\pi}{4}\right) \sim$$

$$\sim \frac{2}{\sqrt{\pi}}\sum_{n=0}^{\infty}(-1)^n \frac{(4n!)}{(2n)!}\left(\frac{1}{2\sqrt{t}}\right)^{4n+1}. \qquad (4.39\text{b})$$

The asymptotic expansions (4.39a) and (4.39b) can be written in the form†

$$\left\{\frac{1}{\sqrt{2}} - C(\sqrt{t})\right\} \cos t + \left\{\frac{1}{\sqrt{2}} - S(\sqrt{t})\right\} \sin t$$

$$= \frac{2}{\sqrt{\pi}}\sum_{n=0}^{N}(-1)^n \frac{(4n+1)!}{(2n)!}\left(\frac{1}{2\sqrt{t}}\right)^{4n+3} - r_N^{(1)}(t) \quad (4.40\text{a})$$

† It should be remembered that in the derivation of the relation (4.39a) we changed the sign in the original asymptotic expansions, which explains the sign in front of the remainder term in (4.40a).

and

$$\left\{\frac{1}{\sqrt{2}}-C(\sqrt{t})\right\}\sin t-\left\{\frac{1}{\sqrt{2}}-S(t)\right\}\cos t$$

$$=\frac{2}{\sqrt{\pi}}\sum_{n=0}^{N}(-1)^n\frac{(4n)!}{(2n)!}\left(\frac{1}{2\sqrt{t}}\right)^{4n+1}+r_N^{(2)}(t), \tag{4.40b}$$

where for any fixed N

$$\lim_{t\to\infty} t^{2N+\frac{3}{2}}r_N^{(1)}(t) = 0 \quad\text{and}\quad \lim_{t\to\infty} t^{2N+\frac{1}{2}}r_N^{(2)}(t) = 0.$$

From each of the relations (4.40a) and (4.40b) it follows, by the way, that as $t \to \infty$, $C(\sqrt{t}) \to 1/\sqrt{2}$ and $S(\sqrt{t}) \to 1/\sqrt{2}$, that is, that

$$\frac{2}{\sqrt{\pi}}\int_0^\infty \cos(u^2)\,du = \frac{1}{\sqrt{2}}, \qquad \frac{2}{\sqrt{\pi}}\int_0^\infty \sin(u^2)\,du = \frac{1}{\sqrt{2}}.$$

Multiplying equation (4.40a) by $\cos t$, and (4.40b) by $\sin t$ and adding them, we find, that

$$\frac{1}{\sqrt{2}}-C(\sqrt{t}) = \frac{2}{\sqrt{\pi}}\cos t\cdot\sum_{n=0}^{N}(-1)^n\frac{(4n+1)!}{(2n)!}\left(\frac{1}{2\sqrt{t}}\right)^{4n+3}+$$

$$+\frac{2}{\sqrt{\pi}}\sin t\cdot\sum_{n=0}^{N}(-1)^n\frac{(4n)!}{(2n)!}\left(\frac{1}{2\sqrt{t}}\right)^{4n+1}-$$

$$-r_N^{(1)}(t)\cos t+r_N^{(2)}(t)\sin t.$$

Replacing in this identity t by t^2, we obtain the final relation:

$$\frac{1}{\sqrt{2}}-C(t) = \frac{2}{\sqrt{\pi}}\cos(t^2)\cdot\sum_{n=0}^{N}(-1)^n\frac{(4n+1)!}{(2n)!}\left(\frac{1}{2t}\right)^{4n+3}+$$

$$+\frac{2}{\sqrt{\pi}}\sin(t^2)\cdot\sum_{n=0}^{N}(-1)^n\frac{(4n)!}{(2n)!}\left(\frac{1}{2t}\right)^{4n+1}+r_N^{(3)}(t), \tag{4.41a}$$

where

$$r_N^{(3)}(t) = -\cos(t^2)r_N^{(1)}(t^2)+\sin(t^2)r_N^{(2)}(t^2).$$

Taking away from equation (4.40a), multiplied by $\sin t$, equation (4.40b) multipled by $\cos t$, and replacing in the result t by t^2, we similarly find that

$$\frac{1}{\sqrt{2}}-S(t) = \frac{2}{\sqrt{\pi}}\sin(t^2)\cdot\sum_{n=0}^{N}(-1)^n\frac{(4n+1)!}{(2n)!}\left(\frac{1}{2t}\right)^{4n+3}-$$

$$-\frac{2}{\sqrt{\pi}}\cos(t^2)\cdot\sum_{n=0}^{N}(-1)^n\frac{(4n)!}{(2n)!}\left(\frac{1}{2t}\right)^{4n+1}+r_N^{(4)}(t), \qquad (4.41\text{b})$$

where

$$r_N^{(4)}(t) = -\sin(t^2)r_N^{(1)}(t^2)-\cos(t^2)r_N^{(2)}(t^2).$$

Formulas (4.41a) and (4.41b) show the complicated way in which $C(t)$ and $S(t)$ tend to their limits, equal to $1/\sqrt{2}$, as $t \to \infty$.

The estimation of the least errors, with which the values of $C(t)$ and $S(t)$ can be approached by formulas (4.41a) and (4.41b), is exactly similar to the one carried out in detail in the preceding example. In the given case it is possible also to establish the signs of $r^{(1)}{}_{N_t}$ and $r^{(2)}{}_{N_t}$ and show, that they do not exceed in absolute value the smallest term of the corresponding expansion.†

24. The general case of the asymptotic expansion of a primary function. The asymptotic expansions of the cylinder functions

The Laplace transform of a primary function may have not one, but several branch points, as happens, for example, for the Laplace transforms of the cylinder functions (see example 6, Art. 19). In this case the asymptotic expansion of the primary function can be obtained from the inversion formula (3.40) from considerations precisely similar to the ones used in the preceding article in the proof of theorem 11. This theorem can be generalized in the following way.

THEOREM 11.′ *Let $f^*(p)$ have the algebraic or logarithmic branch points $w_1, \ldots, w_l (l \geqslant 1)$ and also the singular points $p_1, \ldots p_k$*

† The increase in precision, briefly considered in the preceding example, is also applicable to the given case.

$(k \geqslant 1)$, *which are not branch points,*† *where* $\operatorname{Im} p_\kappa \neq \operatorname{Im} w_\lambda (\kappa = 1, \ldots, k; \lambda = 1, \ldots, l)$ *and* $\operatorname{Im} w_\lambda \neq \operatorname{Im} w_{\lambda}'$, *if* $\lambda \neq \lambda'$.‡ *Let us assume also, that* $f^*(p)$ *satisfies the conditions of theorem* 7

$$(\text{for } s_0 = \max \{Re\ p_1, \ldots, Re\, p_k, Re\, w_1, \ldots, Re\, w_e)$$

and that, in addition, the condition

$$f^*(p) \to 0 \quad \text{as} \quad |p| \to \infty, \quad \operatorname{Re} p \leqslant s_0' (s_0' > s_0).$$

is satisfied. Finally, let the expansion

$$f^*(p) = q_\lambda^{\gamma_\lambda} \sum_{n=0}^{\infty} c_n^{(\lambda)} q_\lambda^{n r_\lambda} \qquad (\lambda = 1, \ldots, l), \tag{4.42}$$

hold in the neighbourhood of every point w_λ, *where* $q_\lambda = p - w_\lambda$ *and* $q_\lambda^\alpha = e^{\alpha \ln q_\lambda} (-\pi < \operatorname{Im} \ln q_\lambda \leqslant \pi)$, *and be convergent for* $|q_\lambda| < \rho_\lambda$.§ *Then, if*

$$f^*(p) = \mathbf{L} f(t)$$

$$f(t) - \sum_{\kappa=1}^{k} \operatorname{res} f^*(p_\kappa) e^{p_\kappa t} = \sum_{\lambda=1}^{l} e^{w_\lambda} f_\lambda(t), \tag{4.43}$$

where

$$f_\lambda(t) \sim \sum_{n=0}^{\infty} \frac{c_n^{(\lambda)}}{\Gamma(-\gamma_\lambda - n r_\lambda)} t^{-\gamma_\lambda - n r_\lambda - 1}. \tag{4.44}$$

Let us note, that theorem 11 is obviously the particular case $l = 1$, $w_1 = 0$ of theorem 11′.

Proceeding to the proof of theorem 11′, let us note, that $f^*(p)$ is regular (not counting the points $p_\kappa (\kappa = 1, \ldots, k)$) in the p-plane

† If the singular points p_κ (in the neighbourhood of which $f^*(h)$ is single valued) are absent, then in the following formulas the sum of the residues of the integrand in the inversion formula of the Laplace transformation will be absent.

‡ These restrictions on the imaginary parts of p_κ and w_λ are not essential and can easily be removed. We applied them only for the purpose of simplifying the discussion.

§ The radius of convergence ρ_λ is, of course, equal to the distance from w_λ to the nearest singular point of the function $f^*(p)$, which is distinct from w_λ.

cut along the rays, issuing from the points $w_\lambda(\lambda = 1, \ldots, l)$ parallel to the real axis in the direction of the negative semi-axis. Let us denote by $H^{(\lambda)}(\lambda = 1, \ldots, l)$ the path, which consists of the segment $\operatorname{Re} p < \operatorname{Re} w_\lambda - \epsilon_\lambda$ of the lower edge of the cut, issuing from the point w_λ, the circle of radius $\epsilon_\lambda < \rho_\lambda$ with centre at the point w_λ, which goes round this point in the positive sense, and the segment $\operatorname{Re} w_\lambda - \epsilon_\lambda > \operatorname{Re} p$ of the upper edge of the cut (Fig. 22). Also, let us denote by K_R the set of arcs of the circle

$$|p| = R = \sqrt{(s^2 \times l^2)} > \max\{|p_1|, \ldots, |p_k|, |w_1|, \ldots, |w_l|\},$$

which connect the point $s+ib$ of the path L to the point $s-ib$ through the left hand half-plane and are broken at the points of intersection of this circle with the paths $H^{(\lambda)}(\lambda = 1, \ldots, l)$. Finally, let us denote by $H_R{}^{(\lambda)}(\lambda = 1, \ldots, l)$ the part of the path $H^{(\lambda)}$ which lies inside the circle $|p| = R$ and is oriented in the opposite direction. Now applying the residue theorem to the contour C_b, which consists of the contour $L_b(-b \leqslant \sigma \leqslant b)$ of the path L, the arc K_R and the paths $H_R{}^{(\lambda)}(\lambda = 1, \ldots l)$, we find, that

$$\frac{1}{2\pi i} \oint_{C_b} f^*(p)e^{pt}\mathrm{d}p = \sum_{\kappa=1}^{k} \operatorname{res} f^*(p_\kappa)e^{p_\kappa t},$$

and consequently,

$$f(t) = \lim_{b\to\infty} \frac{1}{2\pi i} \int_{L_s} f^*(p)e^{pt}\mathrm{d}p =$$

$$= \lim_{b\to\infty} \left\{ \frac{1}{2\pi i} \oint_{C_b} f^*(p)e^{pt}\mathrm{d}p - \frac{1}{2\pi i} \int_{K_R} f^*(p)e^{pt}\mathrm{d}p - \right.$$

$$\left. - \sum_{\lambda=1}^{l} \frac{1}{2\pi i} \int_{H_R{}^{(\lambda)}} f^*(p)e^{pt}\mathrm{d}p \right\} = \sum_{\kappa=1}^{k} \operatorname{res} f^*(p_\kappa)e^{p_\kappa t} -$$

$$- \lim_{b\to\infty} \frac{1}{2\pi i} \int_{K_R} f^*(p)e^{pt}\mathrm{d}p - \sum_{\lambda=1}^{t} \frac{1}{2\pi i} \int_{H^{(\lambda)}} f^*(p)e^{pt}\mathrm{d}p. \tag{4.45}$$

But as in the proof of theorem 10 (Art. 22) it can be shown, that with the given assumptions

$$\lim_{R\to\infty}\frac{1}{2\pi i}\int_{K_R} f^*(p)e^{pt}\mathrm{d}p = 0,$$

and from (4.45) it follows, that

$$f(t)-\sum_{\kappa=1}^{k}\operatorname{res} f^*(p_\kappa)e^{p_\kappa t} = \sum_{\lambda=1}^{l} g_\lambda(t), \tag{4.46}$$

where

$$g_\lambda(t) = \frac{1}{2\pi i}\int_{H^{(\lambda)}} f^*(p)e^{pt}\mathrm{d}p \qquad (\lambda = 1, \ldots, l). \tag{4.47}$$

It now only remains for us to show, that $e^{w_\lambda t}g_\lambda(t)$ has the asymptotic expansion given in (4.44). This is easily done by reducing it to the case, already considered in the preceding article. In fact, putting $p = w_\lambda + q_\lambda$, we can rewrite equation (4.47) in the form

$$g_\lambda(t) = e^{w_\lambda t}\cdot\frac{1}{2\pi i}\int_{H} f^*(w_\lambda+q_\lambda)e^{q_\lambda t}\mathrm{d}q_\lambda, \tag{4.48}$$

where H is the path in the q_λ-plane, cut along the real negative semi-axis, already considered several times above and consisting of the lower edge of the cut from $-\infty$ to the point $-\epsilon_\lambda$, the circle $|q_\lambda| = \epsilon_\lambda$, which goes round the point $q_\lambda = 0$ in the positive direction, and the upper edge of the cut from the point $-\epsilon_\lambda$ to $-\infty$. As (see (4.42))

$$f^*(w_\lambda+q_\lambda) = q_\lambda^{\gamma_\lambda}\sum_{n=0}^{\infty} c_n^{(\lambda)}q_\lambda^{nr_\lambda},$$

it follows that the discussion carried out above for the integral along the path H, which occurs in formula (4.22) (see pages 184–188), can be transferred as a whole to the integral (4.48), and we obtain for the function $e^{-w_\lambda t}-g_\lambda(t) = f_\lambda(t)$ the required asymptotic expansion (4.44), which together with equation (4.43) also proves the theorem.

In using formula (4.43) for the purpose of calculation it is necessary to bear in mind the following remarks.

Remark 1. The relation (4.44) signifies, that

$$f_\lambda(t) = \sum_{n=0}^{N\lambda} \frac{c_n^{(\lambda)}}{\Gamma(-\gamma_\lambda - nr_\lambda)} t^{-\gamma_\lambda - nr_\lambda - 1} + r_{\lambda, N_\lambda}(t),$$

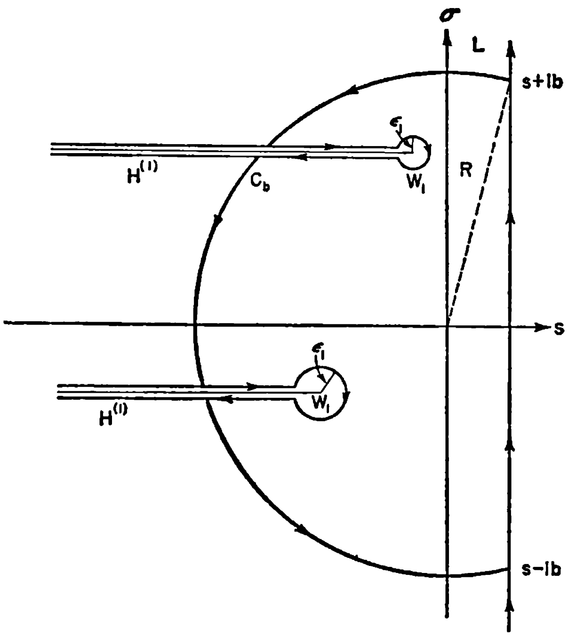


FIG. 22

where for any fixed N_λ

$$\lim_{t\to\infty} t^{\gamma_\lambda + N_\lambda r_\lambda + 1} r_{\lambda, N_\lambda}(t) = 0.$$

Consequently, the relation (4.43) signifies, that

$$f(t) - \sum_{\kappa=1}^{k} \operatorname{res} f^*(p_\kappa) e^p =$$

$$= \sum_{\lambda=1}^{l} \left\{ e^w \sum_{n=0}^{N_\lambda} \frac{c_n^{(\lambda)}}{\Gamma(-\gamma_\lambda - nr_\lambda)} t^{-\gamma_\lambda - nr_\lambda - 1} \right\} + r(t), \tag{4.49}$$

where

$$r(t) = \sum_{\lambda=1}^{l} e^{w_\lambda t} \, . \, r_{\lambda, N_\lambda}(t),$$

and

$$\lim_{t\to\infty} t^a e^{-wt} r(t) = 0,$$

where $a = \min\{\gamma_1+N_1r_1+1, \ldots, \gamma_l+N_lr_l+1\}$ and w denotes that one of the w_λ, which has the greatest real part. In the case where $\operatorname{Re} w_1 = \ldots = \operatorname{Re} e_l$, which is the most important in practice, $N_1, \ldots, N_l$ in formula (4.49) must be selected in such a way that, $b-a$ will be minimal, where $b = \max\{\gamma_1+N_1r_1+1, \ldots, \gamma_l+N_lr_l+1\}$ that is the asymptotic series in (4.44) must be broken off in such a way, that their remainder terms will have as $t \to \infty$ orders of smallness, which differ by as little as possible.

Remark 2. From the representation (4.49) it follows, that for sufficiently great t those terms will dominate in the sum with respect to λ, for which the w_λ have the greatest real part.

The following is also extremely important.

SUPPLEMENT TO THEOREM 11′. When the conditions of theorem 11 are satisfied, the relations (4.43) and (4.44) can be differentiated term by term, that is, it follows from them, that

$$f'(t) - \sum_{\kappa=1}^{k} \operatorname{res} p_\kappa f^*(p_\kappa)e^{p_\kappa t} =$$

$$= \sum_{\lambda=1}^{l} e^{w_\lambda t}\{w_\kappa f_\lambda(t) + f'_\lambda(t)\},$$

where $f_\lambda(t)$ has the asymptotic expansion (4.44), and the asymptotic expansion of $f'_\lambda(t)$ is obtained from it by formal differentiation:

$$f_\lambda(t) \sim \sum_{n=0}^{\infty} \frac{c_n^{(\lambda)}}{\Gamma(-\gamma_\lambda - nr_\lambda - 1)} t^{-\gamma_\lambda - nr_\lambda - 2} \quad (\lambda = 1, \ldots, l). \tag{4.50}$$

In order to prove this proposition, it is obviously, sufficient that the relation (4.50) is true, where by (4.48) (as $g_\lambda(t) = e^{w_\lambda t} f_\lambda(t)$)

$$f'\ (t) = \frac{d}{dt}\frac{1}{2\pi i}\int_H f^*(w_\lambda + q_\lambda)e^{q_\lambda t}dq_\lambda.$$

It is not difficult to see, that differentiation with respect to t can be carried out under the sign of integration,† so that

$$f'_\lambda(t) = \frac{1}{2\pi i}\int\limits_H q_\lambda f^*(w_\lambda+q_\lambda)e^{q_\lambda t}dq_\lambda.$$

But the asymptotic expansion of this integral is obtained in the same way as the expansion (4.44) from the expansion

$$q_\lambda f^*(w_\lambda+q_\lambda) = q_\lambda^{\gamma_\lambda+1}\sum_{n=0}^{\sim} c_n^{(\lambda)}q_\lambda^{nr_\lambda}$$

in the neighbourhood of the point w_λ in the form

$$f'\ (t) \sim \sum_{n=0}^{\infty}\frac{c_n^{(\lambda)}}{\Gamma(-\gamma_\lambda-nr_\lambda-1)}e^{-\gamma_\lambda-nr_\lambda-2}$$

by replacing in (4.44) γ_λ by $\gamma_\lambda+1$, and the relation (4.50) is proved.

Let us now pass on to the application of these results to the derivation of the asymptotic expansions of the cylinder functions. Let us begin with the function $J_0(t)$, for which (see example 6, Art. 19)

$$f^*(p) = \mathrm{L}J_0(t) = \frac{1}{\sqrt{(p^2+1)}}.$$

The conditions of theorem 11′ are satisfied, with $w_1 = -i$ and $w_2 = i$. The function $f^*(p)$ has no other singular points.‡

In the neighbourhood of the point $w_1 = -i$ let us put $p = q_1-i$, so that

$$f^*(p) = \frac{1}{\sqrt{(q_1^2-2iq_1)}} =$$

$$= \frac{1}{\sqrt{(-2i)}}q_1^{-\frac{1}{2}}\left(1-\frac{q_1}{2i}\right)^{-\frac{1}{2}} = \frac{1}{\sqrt{2}}e^{i\pi/4}q_1^{-\frac{1}{2}}\sum_{n=0}^{\infty}\frac{(2n)!}{2^{2n}(n!)^2(2i)^n}q_1^n$$

---

† In view of the uniform convergence of the resulting integral.

‡ In the given case the points p_κ are absent. See footnote‡ on page 218.

and, consequently,

$$f_1(t) \sim \frac{1}{\sqrt{2}} e^{i\pi/4} \sum_{n=0}^{\infty} \frac{(2n)!}{2^{2n}(n!)^2(2i)^n} \frac{1}{\Gamma(-n+\frac{1}{2})} t^{-n-\frac{1}{2}},$$

or, inserting the value of $\Gamma(-n+\frac{1}{2})$,

$$f_1(t) \sim \frac{1}{\sqrt{(2\pi t)}} e^{i\pi/4} \sum_{n=0}^{\infty} (-1)^n \frac{[(2n)!]^2}{2^{4n}(n!)^3(2i)^n} t^{-n}.$$

This asymptotic expansion can be written more conveniently in a somewhat different form, by separating the sum into its real and imaginary parts:

$$f_1(t) = \frac{e^{i\pi/4}}{\sqrt{(2\pi t)}} \{\phi_0(t) + i\psi_0(t)\}.$$

Putting $n = 2m$, we find, that

$$\phi_0(t) \sim \sum_{m=0}^{\infty} (-1)^m \frac{[(4m)!]^2}{2^{10m}[(2m)!]^3} t^{-2m}, \tag{4.51}$$

and putting $n = 2m+1$, that

$$\psi_0(t) \sim \sum_{m=0}^{\infty} (-1)^m \frac{[(4m+2)!]^2}{2^{10m+5}[(2m+1)!]^3} t^{-2m-1}. \tag{4.52}$$

It is not difficult to see, that

$$f_2(t) = \frac{e^{-i\pi/4}}{\sqrt{(2\pi t)}} \{\phi_0(t) - i\psi_0(t)\},$$

and, consequently, by (4.43)

$$J_0(t) = \frac{1}{\sqrt{(2\pi t)}} e^{-i[t-\pi/4]} \{\phi_0(t) + i\psi_0(t)\} +$$

$$+ \frac{1}{\sqrt{(2\pi t)}} e^{i[t-\pi/4]} \{\phi_0(t) - i\psi_0(t)\} =$$

$$= \sqrt{\left(\frac{2}{\pi t}\right)} \phi_0(t) \cos\left(t - \frac{\pi}{4}\right) + \sqrt{\left(\frac{2}{\pi t}\right)} \psi_0(t) \sin\left(t - \frac{\pi}{4}\right), \tag{4.53}$$

where $\phi_0(t)$ and $\psi_0(t)$ have the asymptotic expansions (4.51) and (4.52).

It can be shown (we shall not stop for this), that in the expansions (4.51) and (4.52) the remainder term does not exceed in absolute value the absolute value of the first neglected term of the expansion (see example 1, Art. 23), that is, that

$$\phi_0(t) = \sum_{m=0}^{M} (-1)^m \frac{[(4m)!]^2}{2^{10m}[(2m)!]^3} t^{-2m} + r_M^{(1)}(t),$$

where

$$|r_M^{(1)}(t)| < \frac{[(4M+4)!]^2}{2^{10M+10}[(2M+2)!]^3} t^{-2M-2},$$

and

$$\psi_0(t) = \sum_{m=0}^{M} (-1)^m \frac{[(4m+2)!]^2}{2^{10m+5}[(2m+1)!]^3} t^{-2m-1} + r_M^{(2)}(t),$$

where

$$|r_M^{(2)}(t)| < \frac{[(4M+6)!]^2}{2^{10M+15}[(2M+3)!]^3} t^{-2M-3}.$$

From the relations (4.51), (4.52) and (4.53), it follows, in particular, that

$$\lim_{t\to\infty} \left\{ J_0(t) - \sqrt{\left(\frac{2}{\pi t}\right)} \cos\left(t - \frac{\pi}{4}\right) \right\} = 0,$$

and also, that

$$\lim_{t\to\infty} t \left\{ J_0(t) - \sqrt{\left(\frac{2}{\pi t}\right)} \left[\cos\left(t - \frac{\pi}{4}\right) + \frac{1}{8t} \sin\left(t - \frac{\pi}{4}\right) \right] \right\} = 0,$$

and so on.

In order to obtain the asymptotic expansion of the function $J_\nu(t)$ use can be made of the relation (3.39). However, it is difficult to obtain the expansion of the function

$$\mathbf{L}J_\nu(t) = \frac{[\sqrt{(p^2+1)} - p]^\nu}{\sqrt{(p^2+1)}}$$

in a power series in the neighbourhoods of the branch points $w_1 = -i$ and $w_2 = i$, starting from this representation of the function, if $\nu > 1$, and hence we shall choose a different path.

Let $\mathrm{L}J_\nu(t) = f_\nu^*(p)$. Then $\mathrm{L}J'_\nu(t) = pf^*(p)$ and $\mathrm{L}J_\nu''(t) = p^2f^*(p)$, if $\nu > 1$, and $\mathrm{L}J_1''(t) = p^2f^*(p)-1$. As we have the identity

$$t^2J''_\nu(t)+tJ_\nu'(t)+(t^2-\nu^2)J_\nu(t) = 0,$$

for $\nu = 1$ we find by (3.5), that

$$\frac{\mathrm{d}^2}{\mathrm{d}p^2}\{p^2f_1^*(p)-1\}-\frac{\mathrm{d}}{\mathrm{d}p}\{pf_1^*(p)\}+\frac{\mathrm{d}^2}{\mathrm{d}p^2}f_1^*(p)-f_1^*(p) = 0,$$

and for $\nu > 1$, that

$$\frac{\mathrm{d}^2}{\mathrm{d}p^2}\{p^2f_\nu^*(p)\}-\frac{\mathrm{d}}{\mathrm{d}p}\{pf_\nu^*(p)\}+$$

$$+\frac{\mathrm{d}^2}{\mathrm{d}p^2}f_\nu^*(p)-\nu^2f_\nu^*(p) = 0. \tag{4.54}$$

As the preceding equation is, obviously, a particular case of equation (4.54) for $\nu = 1$, it follows that the latter is true for $\nu = 1$ also, that is for all integral $\nu \geqslant 1$ (in fact, as is not difficult to see, for $\nu = 0$ also). Equation (4.54) can be written in the form

$$(p^2+1)f_\nu^{*\prime\prime}(p)+3pf_\nu^{*\prime}(p)-(\nu^2-1)f_\nu^*(p) = 0.$$

It is, of course, satisfied by the function (3.39). Putting $p = q_1-1$ and $f_\nu^*(p) = y_\nu^*(q_1)$, we transform this equation into the form

$$(q_1^2-2iq_1)\frac{\mathrm{d}^2y_\nu^*}{\mathrm{d}q_1^2}+3(q_1-i)\frac{\mathrm{d}y_\nu^*}{\mathrm{d}q_1}-(\nu^2-1)y_\nu^* = 0. \tag{4.55}$$

From formula (3.39)) we conclude, that in the neighbourhood of the point $p = w_1 = -i$ an expansion of the form†

$$f_\nu^*(p) = y_\nu^*(q_1) = q_1^{-\frac{1}{2}}\sum_{n=0}^{\infty} c_nq_1^n = \sum_{n=0}^{\infty} c_nq_1^{n-\frac{1}{2}}, \tag{4.56}$$

must hold, so that

$$\frac{\mathrm{d}y_\nu^*}{\mathrm{d}q_1} = \sum_{n=0}^{\infty}(n-\tfrac{1}{2})c_nq_1^{n-\frac{3}{2}}$$

† Of course the coefficients c_n depend also on ν, but we do not express this in the notation in order not to complicate the formulas.

and

$$\frac{d^2y_\nu^*}{dq_1^2} = \sum_{n=0}^{\infty} (n-\tfrac{1}{2})(n-\tfrac{3}{2})c_n q_1^{n-\frac{5}{2}}.$$

Substituting this expansion in (4.55), we obtain the identity

$$\sum_{n=0}^{\infty} \{(n-\tfrac{1}{2})(n-\tfrac{3}{2})+3(n-\tfrac{1}{2})-(\nu^2-1)\}c_n q_1^{n-\frac{3}{2}}+$$

$$+\sum_{n=0}^{\infty} \{-2i(n-\tfrac{1}{2})(n-\tfrac{3}{2})-3i(n-\tfrac{1}{2})\}c_n q_1^{n-\frac{5}{2}} = 0.$$

In the second of these sums the term, which corresponds to $n = 0$, becomes zero, so that in this sum n can be considered as running through the values from 1 to ∞. By then replacing in it n by $n+1$, we can combine both sums and obtain the identity

$$\sum_{n=0}^{\infty} [\{(n-\tfrac{1}{2})(n-\tfrac{3}{2})+3(n-\tfrac{1}{2})-(\nu^2-1)\}c_n+$$

$$+\{-2i(n+\tfrac{1}{2})(n-\tfrac{1}{2})-3i(n+\tfrac{1}{2})\}c_{n+1}]q_1^{n-\frac{3}{2}} = 0,$$

or

$$\sum_{n=0}^{\infty} [\{(n+\tfrac{1}{2})^2-\nu^2\}c_n-(i/2)(2n+1)(2n+2)c_{n+1}]q_1^{n-\frac{3}{2}} = 0,$$

whence it follows, that

$$c_{n+1} = \frac{1}{2i}\frac{(2n+1)^2-4\nu^2}{(2n+1)(2n+2)}c_n \qquad (n = 0,\ 1,\ \ldots). \tag{4.57}$$

Starting from the recurrence relation (4.57), it is easy to obtain the expression for the coefficients $c_n (n = 1, 2, \ldots)$. In fact, putting in (4.57) $n = 0$, we find, that

$$c_1 = \frac{1}{2i}\frac{1^2-4\nu^2}{2!}c_0;$$

putting $n = 1$, we find that

$$c_2 = \frac{1}{2i}\frac{3^2-4\nu^2}{3\,.\,4}c_1 = \left(\frac{1}{2i}\right)^2\frac{(1^2-4\nu^2)(3^2-4\nu^2)}{4!}c_0,$$

and so on. In general

$$c_n = \left(\frac{1}{2i}\right)^n\frac{(1^2-4\nu^2)\,\ldots\,\{(2n-1)^2-4\nu^2\}}{(2n)!}c_0 \qquad (n = 1,\ 2,\ \ldots). \quad (4.58)$$

As regards c_0, this coefficient is found from the representation (3.39), by considering that

$$c_0 = \lim_{q_1\to 0} q_1^{\frac{1}{2}}y_\nu^*(q_1).$$

As

$$y_\nu^*(q_1) = \frac{[\sqrt{(q_1^2-2iq_1)}-q_1+i]^\nu}{\sqrt{(q_1^2-2iq_1)}},$$

it follows from this that

$$c_0 = \frac{i^\nu}{\sqrt{(-2i)}} = \frac{i^\nu}{\sqrt{2}}e^{i(\pi/4)}$$

and by formula (4.58)

$$c_n = \frac{i^\nu}{\sqrt{2}}e^{(i\pi/4)}\left(\frac{1}{2i}\right)^n\frac{(1^2-4\nu^2)\,\ldots\,\{(2n-1)^2-4\nu^2\}}{(2n)!} \qquad (n = 1,\ 2,\ \ldots).$$

By the expansions (4.44) and (4.56) (compare with formula (4.42)) we now find the following asymptotic expansion:

$$f_1(t) \sim \frac{i^\nu e^{i_t\pi/4}}{\sqrt{(2)}\Gamma(\frac{1}{2})}t^{-\frac{1}{2}} +$$

$$+\sum_{n=1}^{\infty}\frac{i^\nu}{\sqrt{2}}e^{i\pi/4}\left(\frac{1}{2i}\right)^n\frac{(1^2-4\nu^2)\,\ldots\,\{(2n-1)^2-4\nu^2\}}{(2n)!\Gamma(-n+\frac{1}{2})}t^{-n-\frac{1}{2}},$$

or

$$f_1(t) \sim \frac{1}{\sqrt{(2\pi t)}}i^\nu e^{i\pi/4}\left\{1+\right.$$

$$\left.+\sum_{n=1}^{\infty}(-1)^n\left(\frac{1}{2i}\right)^n\frac{(1^2-4\nu^2)\,\ldots\,\{(2n-1)^2-4\nu^2\}}{2^{2n}n!}t^{-n}\right\},$$

or, finally,

$$f_1(t) = \frac{1}{\sqrt{2\pi t}} e^{i(\nu\pi/2+\pi/4)}\{\phi_\nu(t)+i\psi_\nu(t)\}, \tag{4.59}$$

where $(n = 2m)$

$$\phi_\nu(t) \sim 1+\sum_{m=1}^{\infty}(-1)^m \frac{(1^2-4\nu^2)\dots\{(4m-1)^2-4\nu^2\}}{2^{6m}(2m)!}t^{-2m} \tag{4.60}$$

and $(n = 2m+1)$

$$\psi_\nu(t) \sim \sum_{m=0}^{\infty}(-1)^m \frac{(1^2-4\nu^2)\dots\{(4m+1)^2-4\nu^2\}}{2^{6m+3}(2m+1)!}t^{-2m-1}. \tag{4.61}$$

The asymptotic expansion of the function $f_2(p)$, which corresponds to the branch point $w_2 = i$, is obtained from the representation (4.59) on replacing i by $-i$:†

$$f_2(t) = \frac{1}{\sqrt{(2\pi t)}} e^{-i(\nu\pi/2+\pi/4)}\{\phi_\nu(t)-i\psi_\nu(t)\}.$$

Consequently, taking into account also the regularity of the function $f^*(p) = (\sqrt{(p^2+1)}-p)^\nu/\sqrt{(p^2+1)})$ in the p-plane, cut in the left hand half-plane along the rays, which issue from the points $w_1 = -i$ and $w_2 = i$ parallel to the real axis, by formula (4.43) we finally find, that

$$\begin{aligned} J_\nu(t) &= e^{-it}f_1(t)+e^{it}f_2(t) = \\ &= \frac{1}{\sqrt{(2\pi t)}} e^{-i(t-\nu\pi/2-\pi/4)}\{\phi_\nu(t)+i\psi_\nu(t)+ \\ &\qquad + \frac{1}{\sqrt{(2\pi t)}} e^{i(t-\nu\pi/2-\pi/4)}\{\phi_\nu(t)-i\psi_\nu(t)\} = \\ &= \sqrt{\left(\frac{2}{\pi t}\right)}\phi_\nu(t)\cos\left(t-\nu\frac{\pi}{2}-\frac{\pi}{4}\right)+ \\ &\qquad + \sqrt{\left(\frac{2}{\pi t}\right)}\psi_\nu(t)\sin\left(t-\nu\frac{\pi}{2}-\frac{\pi}{4}\right), \end{aligned} \tag{4.62}$$

where $\phi_\nu(t)$ and $\psi_\nu(t)$ have the asymptotic expansions (4.60) and (4.61).

† As is easily seen, beginning with the expansion (4.56), in which it is necessary to replace q_1 by q_2, where $p = q_2+i$, and c_n by $\bar{c}_n$.

From formula (4.62) together with the expansions (4.60) and (4.61), it follows, in particular, that

$$\lim_{t\to\infty}\left\{J_\nu(t)-\sqrt{\left(\frac{2}{\pi t}\right)}\cos\left(t-\nu\frac{\pi}{2}-\frac{\pi}{4}\right)\right\}=0,$$

and also, that

$$\lim_{t\to\infty} t\left\{J_\nu(t)-\sqrt{\left(\frac{2}{\pi t}\right)}\left[\cos\left(t-\nu\frac{\pi}{2}-\frac{\pi}{4}\right)+\right.\right.$$
$$\left.\left.+\frac{1-4\nu^2}{8t}\sin\left(t-\nu\frac{\pi}{2}-\frac{\pi}{4}\right)\right]\right\}=0,$$

and so on.

Above (page 225) we noted the particular case $\nu = 0$ of these relations. In general formula (4.53) is contained as the particular case $\nu = 0$ in formula (4.62). This is easy to see from the comparison of the asymptotic expansions (4.60) and (4.61) with (4.51) and (4.52).†

There is no difficulty in carrying out the estimation of the remainder terms of the expansions (4.60) and (4.61) and finding, that these remainder terms do not exceed in their absolute value the absolute value of the first neglected term of the expansion.

Before concluding the consideration of the cylinder functions, let us also make the following remark.

The inversion formula (3.40) is the *integral representation* of the primary function $f(t)$, which, as a rule, can also be transformed by deformation of the path of integration, and sometimes also by changing the variable. Such integral representations are extremely useful in the study of special functions and frequently make it possible to discover their deeply lying properties, which it is difficult to find by any other method.

In particular, the inversion formula (3.40) can be applied to the cylinder function $J_\nu(t)$, for which (see (3.39))

$$f^*(p)=\frac{[\sqrt{(p^2+1)}-p)^\nu}{\sqrt{(p^2+1)}},$$

† The asymptotic expansions (4.62) and (4.60), (4.61) can also be obtained by a different method, namely by the application of theorem 11′ (on the permissibility of the term by term differentiation of asymptotic expansions of the type considered) to the recurrence relations between cylinder functions (of the type (3.38)). It is possible to find the asymptotic expansion of $J_1(t)$, starting from the formula $J_1(t) = -J'_0(t)$. However, the method chosen by us is more convenient.

and we find, that

$$J_\nu(t) = \frac{1}{2\pi i}\int\limits_L \frac{[\sqrt{(p^2+1)}-p]^\nu}{\sqrt{(p^2+1)}} e^{pt} dp,$$

where L is the path $\text{Re}\, p = s > 0$.† The integrand is regular in the p-plane with the corresponding cuts. We shall now replace the cuts from the branch points $w_1 = -i$ and $w_2 = i$ parallel to the real axis in the left hand half-plane made by us above in considering asymptotic expansions, by a single rectilinear cut, connecting these points (Fig. 23), where at a point on the right hand edge of the cut $\sqrt{(p^2+1)} = 1$, and at the point $p = 0$ on the left hand edge of the cut $\sqrt{(p^2+1)} = -1$. As the points $w_1 = -i$ and $w_2 = i$ are integrable singularities (the integrand increases sufficiently slowly as it is approached), the path L can be traversed along the imaginary axis itself, but on the segment of the latter from $-i$ to i it must, of course, be traversed along the right hand edge of the cut.

Let us consider the contour, which consists of the segment of the imaginary axis from the point Ri to the point $-Ri (R > 1)$, which includes the left hand edge of the cut and the semicircle $K_R : |p| = R$, $\text{Re}\, p \leqslant 0$. By Cauchy's theorem, because of the regularity of the integrand within it, the integral along this contour will be equal to zero, and the integral along K_R will tend to zero as $R \to \infty$ (as was shown above). Consequently, the integral along the path L', which is represented by the imaginary axis traversed from $i\infty$ to $-i\infty$, including the left hand edge of the cut (Fig. 23) is equal to zero. Thus

$$J_\nu(t) = \frac{1}{2\pi i}\int\limits_L \frac{[\sqrt{(p^2+1)}-p]^\nu}{\sqrt{(p^2+1)}} e^{pt} dp + \frac{1}{2\pi i}\int\limits_{L'} \frac{[\sqrt{(p^2+1)}-p]^\nu}{\sqrt{(p^2+1)}} e^{pt} dp =$$

$$= \frac{1}{2\pi i}\oint\limits_0 \frac{[\sqrt{(p^2+1)}-p]^\nu}{\sqrt{(p^2+1)}} e^{pt} dp,$$

† Although the preceding results were derived on the supposition, that ν is a negative integer, they are correct for all $\nu > -\frac{1}{2}$. However, the integral representation of the cylinder functions, to the derivation of which we are proceeding, is correct only for $\nu = 0, 1, 2, \ldots$

where 0 is a closed path, which consists of the edges of the cut and goes round the branch points $w_1 = -i$ and $w_2 = i$ in the positive direction. This follows from the fact that the integral along the parts of L and L' which are not edges of the cuts, cancel one another out.

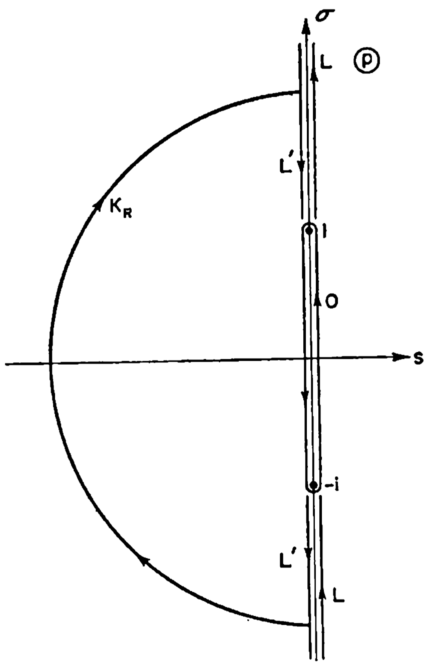


FIG. 23

Let us now put $p = i \sin \zeta$, where $\zeta = \phi + i\psi$. Then

$$\frac{[\sqrt{(p^2+1)}-p]^\nu}{\sqrt{(p^2+1)}} e^{pt} \mathrm{d}p =$$

$$= [\sqrt{(1-\sin^2\zeta)}-i\zeta]^\nu e^{it \sin \zeta} i \mathrm{d}\zeta = i e^{i(t \sin \zeta - i\zeta)} \mathrm{d}\zeta,$$

where at the point $p = 0$ of the right hand edge of the cut

$$\sqrt{(p^2+1)} = \cos \zeta = 1,$$

that is, to this point corresponds the point $\zeta = 0$, and at the point $p = 0$ of the left hand edge of the cut $\sqrt{(p^2+1)} = \cos \zeta = -1$, that is, to this point correspond the points $\zeta = \pm\pi$. Consequently,

in the ζ-plane the path 0 passes into the path $\psi = 0$, $-\pi \leqslant \phi \leqslant \pi$, and we find, that

$$J_\nu(t) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i(t\sin\phi - \nu\phi)}\mathrm{d}\phi = \frac{1}{2\pi}\int_{-\pi}^{\pi} \cos\{t\sin\phi - \nu\phi\}\mathrm{d}\phi =$$

$$= \frac{1}{\pi}\int_{0}^{\pi} \cos\{t\sin\phi - \nu\phi\}\mathrm{d}\phi.$$

There also exist other integral representations of the cylinder functions, which can also be obtained from the inversion formula (3.40), but we shall limit ourselves to the derivation of the integral representation, which is called Bessel's integral.

25. The saddle point method. The asymptotic expansion of the gamma function

By this method it is possible to obtain the first terms of an asymptotic expansion (and in some cases even the whole expansion) as $\tau \to \infty$ of the integral

$$\int_L a(p)e^{\tau b(p)}\mathrm{d}p,$$

if the functions $a(p)$, $b(p)$ and the path L satisfy certain extremely general conditions. This method is based on the following proposition, which in its own right plays an important part in mathematical analysis.

LEMMA. *Let $\beta(\lambda)$ be a real function of the real variable $\lambda(-\infty < \lambda < \infty)$, which assumes its greatest value for $\lambda = 0$; $\beta(\lambda) < \beta(0) = \beta_0$ for $|\lambda| > 0$, and in any interval ($\epsilon > 0$), $-\infty < \lambda < -\epsilon$ and $\epsilon < \lambda < \infty$, $\beta(\lambda) < \beta_0 - \delta$, $\delta = \delta(\epsilon) > 0$. Let, also, $\alpha(\lambda)$ be a function (generally speaking, complex-valued) of λ, such that*

$$\int_{-\infty}^{\infty} |\alpha(\lambda)| e^{\tau_0 \beta(\lambda)}\mathrm{d}\lambda = A < \infty,$$

where $\tau_0 > 0$. *Then, if in a certain neighbourhood of the point* $\lambda = 0$ *the expansions*

$$\alpha(\lambda) = \alpha_0 + \alpha_1\lambda + \alpha_2\lambda^2 + \dots$$

and

$$\beta(\lambda) = \beta_0 + \beta_2\lambda^2 + \beta_3\lambda^3 + \beta_4\lambda^4 + \dots,$$

where $\beta_2 < 0$ *hold, then*

$$e^{-\tau\beta_0}\int_{-\infty}^{\infty}\alpha(\lambda)e^{\tau\beta(\lambda)}\mathrm{d}\lambda \sim \sqrt{\left(-\frac{\pi}{\beta_2}\right)}\alpha_0\tau^{-\frac{1}{2}} +$$

$$+\sqrt{\left(-\frac{\pi}{\beta_2}\right)}\left\{\frac{\alpha_2}{-2\beta_2} + \frac{3(\alpha_1\beta_3+\alpha_0\beta_4)}{4\beta_2^2} + \frac{15\alpha_0\beta_3^2}{-16\beta_2^3}\right\}\tau^{-\frac{3}{2}} + \dots, \tag{4.63}$$

that is

$$\lim_{\lambda\to\infty}\tau^{\frac{1}{2}}\left\{e^{-\tau\beta_0}\int_{-\infty}^{\infty}\alpha(\lambda)e^{\tau\beta(\lambda)}\mathrm{d}\lambda - \sqrt{\left(-\frac{\pi}{\beta_2}\right)}\alpha_0\tau^{-\frac{1}{2}}\right\} = 0, \tag{4.64a}$$

or

$$\lim_{\lambda\to\infty}\tau^{\frac{1}{2}}e^{-\tau\beta_0}\int_{-\infty}^{\infty}\alpha(\lambda)e^{\tau\beta(\lambda)}\mathrm{d}\lambda = \sqrt{\left(-\frac{\pi}{\beta_2}\right)}\alpha_0, \tag{4.64b}$$

and also

$$\lim_{\lambda\to\infty}\tau^{\frac{3}{2}}\left[e^{-\tau\beta_0}\int_{-\infty}^{\infty}\alpha(\lambda)e^{\tau\beta(\lambda)}\mathrm{d}\lambda - \sqrt{\left(-\frac{\pi}{\beta_2}\right)}\alpha_0\tau^{-\frac{1}{2}} - \right.$$

$$\left. - \sqrt{\left(-\frac{\pi}{\beta_2}\right)}\left\{\frac{\alpha_2}{-2\beta_2} + \frac{3(\alpha_1\beta_3+\alpha_0\beta_4)}{4\beta_2^2} + \frac{15\alpha_0\beta_3^2}{-16\beta_2^3}\right\}\tau^{-\frac{3}{2}}\right] = 0,$$

or

$$\lim_{\tau\to\infty}\left[\tau^{\frac{3}{2}}e^{-\tau\beta_0}\int_{-\infty}^{\infty}\alpha(\lambda)e^{\tau\beta(\lambda)}d\lambda - \sqrt{-\frac{\pi}{\beta_2}}\alpha_0\tau\right] =$$

$$= \sqrt{\left(-\frac{\pi}{\beta_2}\right)}\left\{\frac{\alpha_2}{-2\beta_2} + \frac{3(\alpha_1\beta_3+\alpha_0\beta_4)}{4\beta_2^2} + \frac{15\alpha_0\beta_3^2}{-16\beta_2^3}\right\}.$$

In order to prove these relations let us note in the first place, that (assuming $\tau > \tau_0$)

$$\left|\int\limits_{-\infty}^{-\epsilon} \alpha(\lambda)e^{\tau\{\beta(\lambda)-\beta_0\}}d\lambda\right| =$$

$$= \left|\int\limits_{-\infty}^{-\epsilon} \alpha(\lambda)e^{\tau_0\{\beta(\lambda)-\beta_0\}} \,.\, e^{(\tau-\tau_0)\{\beta(\lambda)-\beta_0\}}d\lambda\right| <$$

$$< e^{-(\tau-\tau_0)\delta}e^{-\tau_0\beta_0}\int\limits_{-\infty}^{-\epsilon} |\alpha(\lambda)|e^{\tau_0\beta(\lambda)}d\lambda < Ae^{-\tau\delta+\tau_0(\delta-\beta_0)},$$

and similarly

$$\left|\int\limits_{\epsilon}^{\infty} \alpha(\lambda)e^{\tau\{\beta(\lambda)-\beta_0\}}d\lambda\right| < Ae^{-\tau\delta+\tau_0(\delta-\beta_0)},$$

and consequently, these integrals have no effect† on the asymptotic expansion of the integral

$$\int\limits_{-\infty}^{\infty} \alpha(\lambda)e^{\tau\beta(\lambda)}d\lambda$$

considered, that is, it is sufficient to consider the integral

$$\int\limits_{-\epsilon}^{\epsilon} \alpha(\lambda)e^{\tau\beta(\lambda)}d\lambda$$

for any sufficiently small $\epsilon > 0$. Let us choose ϵ so small, that in the interval $-\epsilon \leqslant \lambda \leqslant \epsilon$ the representations

$$\alpha(\lambda) = \alpha_0+\alpha_1\lambda+\alpha_2\lambda^2+\alpha_3\lambda^3$$

and

$$\beta(\lambda) = \beta_0+\beta_2\lambda^2+\beta_3\lambda^3+\beta_4\lambda^4+\beta_5\lambda^5,$$

† See the footnote on page 197.

hold, where $\tilde{\alpha}_3 = \tilde{\alpha}_3(\lambda)$ and $\tilde{\beta}_3 = \tilde{\beta}_3(\lambda)$ are bounded for $-\epsilon \leqslant \lambda \leqslant \epsilon$. Then

$$\int_{-\epsilon}^{\epsilon} \alpha(\lambda) e^{\tau\{\beta_0(\lambda)-\beta\}} d\lambda = \int_{-\epsilon}^{\epsilon} \{\alpha_0 + \alpha_1\lambda + \alpha_2\lambda^2 + \tilde{\alpha}_3\lambda^3\} e^{\beta_2\tau\lambda^2 + \beta_3\tau\lambda^3 + \beta_4\tau\lambda^4 + \tilde{\beta}_5\tau\lambda^5} d\lambda$$

or, putting $\omega = \sqrt{(-\beta_2\tau)}\lambda$

$$\int_{-\epsilon}^{\epsilon} \alpha(\lambda) e^{\tau\{\beta(\lambda)-\beta_0\}} d\lambda = \frac{1}{\sqrt{(-\beta_2\tau)}} \int_{-\sqrt{(-\beta_2\tau)}\epsilon}^{\sqrt{(-\beta_2\tau)}\epsilon} \left\{\alpha_0 + \frac{\alpha_1\omega}{\sqrt{(-\beta_2\tau)}} + \frac{\alpha_2\omega^2}{[\sqrt{(-\beta_2\tau)}^2} + \right.$$

$$\left. + \frac{\tilde{\alpha}_3\omega^3}{[\sqrt{(-\beta_2\tau)}]^3}\right\} \exp\left[-\omega^2 + \frac{\beta_3\omega^3}{-\beta_2\sqrt{(-\beta_2\tau)}} + \frac{\beta_4\omega^4}{-\beta_2[\sqrt{(-\beta_2\tau)}]^2} + \right.$$

$$\left. + \frac{\tilde{\beta}_5\omega^5}{-\beta_2[\sqrt{(-\beta_2\tau)}]^3}\right] d\omega.$$

But, expanding the integrand in powers of $\tau^{-\frac{1}{2}}$, we find, that

$$\left\{\alpha_0 + \frac{\alpha_1\omega}{\sqrt{(-\beta_2\tau)}} + \frac{\alpha_2\omega^2}{[\sqrt{(-\beta_2\tau)^2}]} + \ldots\right\}$$

$$\exp\left[\frac{\beta_3\omega^3}{-\beta_2\sqrt{(-\beta_2\tau)}} + \frac{\beta_4\omega^4}{-\beta_2[\sqrt{(-\beta_2\tau)}]^2} + \ldots\right] =$$

$$= \left\{\alpha_0 + \frac{\alpha_1\omega}{\sqrt{(-\beta_2\tau)}} + \frac{\alpha_2\omega^2}{[\sqrt{(-\beta_2\tau)}]^2} + \ldots\right\}$$

$$\left\{1 + \frac{\beta_3\omega^3}{-\beta_2\sqrt{(-\beta_2\tau)}} + \frac{\beta_4\omega^4}{-\beta_2[\sqrt{(-\beta_2\tau)}]^2} + \frac{\beta_3{}^2\omega^6}{2\beta_2{}^2[\sqrt{(-\beta_2\tau)}]^2} + \ldots\right\} =$$

$$= \alpha_0 + \left\{\frac{\alpha_1\omega}{\sqrt{(-\beta_2)}} + \frac{\alpha_0\beta_3\omega^3}{-\beta_2\sqrt{(-\beta_2)}}\right\}\tau^{-\frac{1}{2}} +$$

$$+ \left\{\frac{\alpha_2\omega^2}{-\beta_2} + \frac{\alpha_1\beta_3\omega^4}{\beta_2} + \frac{\alpha_0\beta_4\omega^4}{\beta_2{}^2} + \frac{\alpha_0\beta_3{}^2\omega^6}{-2\beta_2{}^3}\right\}\tau^{-1} + \ldots, \quad (4.65)$$

where the terms of higher orders in $\tau^{-\frac{1}{2}}$ have been omitted. Consequently,

$$\int_{-\epsilon}^{\epsilon}\alpha(\lambda)e^{\tau\{\beta(\lambda)-\beta_0\}}\mathrm{d}\lambda=\left\{\frac{\alpha_0}{\sqrt{(-\beta_2)}}\int_{-\sqrt{(-\beta_2\tau)}\epsilon}^{\sqrt{(-\beta_2\tau)}\epsilon}e^{-\omega^2}\mathrm{d}\omega\right\}\tau^{-\frac{1}{2}}+$$

$$+\left\{\frac{\alpha_1}{-\beta_2}\int_{-\sqrt{(-\beta_2\tau)}\epsilon}^{\sqrt{-(\beta_2\tau)}\epsilon}\omega e^{-\omega^2}\mathrm{d}\omega+\frac{\alpha_0\beta_3}{\beta_2{}^2}\int_{-\sqrt{(-\beta_2\tau)}\epsilon}^{\sqrt{(-\beta_2\tau)}\epsilon}\omega^3e^{-\omega^2}\mathrm{d}\omega\right\}\tau^{-1}+$$

$$+\left\{\frac{\alpha_2}{-\beta_2\sqrt{(-\beta_2)}}\int_{-\sqrt{(-\beta_2\tau)}\epsilon}^{\sqrt{(-\beta_2\tau)}\epsilon}\omega^2e^{-\omega^2}\mathrm{d}\omega+\frac{\alpha_1\beta_3+\alpha_0\beta_4}{\beta_2{}^2\sqrt{(-\beta_2)}}\int_{-\sqrt{(-\beta_2\tau)}\epsilon}^{\sqrt{(-\beta_2\tau)}\epsilon}\omega^4e^{-\omega^2}\mathrm{d}\omega+\right.$$

$$\left.+\frac{\alpha_0\beta_3{}^2}{-2\beta_2{}^3\sqrt{(-\beta_2)}}\int_{-\sqrt{(-\beta_2\tau)}\epsilon}^{\sqrt{(-\beta_2\tau)}\epsilon}\omega^4e^{-\omega^2}\mathrm{d}\omega\right\}\tau^{-\frac{3}{2}}+r(\tau),$$

where

$$\lim_{\tau\to\infty}\tau^{\frac{3}{2}}r(\tau)=0.$$

But in this relation the coefficient of τ^{-1} on the right hand side is equal to zero because of the oddness of the integrand. Also it is obvious, that this relation can be written in the form

$$\int_{-\epsilon}^{\epsilon}\alpha(\lambda)e^{\tau\{\beta(\lambda)-\beta_0\}}\mathrm{d}\lambda=\left\{\frac{\alpha_0}{\sqrt{(-\beta_2)}}\int_{-\infty}^{\infty}e^{-\omega^2}\mathrm{d}\omega\right\}\tau^{-\frac{1}{2}}+$$

$$+\left\{\frac{\alpha_2}{-\beta_2\sqrt{(-\beta_2)}}\int_{-\infty}^{\infty}\omega^2e^{-\omega^2}\mathrm{d}\omega+\frac{\alpha_1\beta_3+\alpha_0\beta_4}{\beta_2{}^2\sqrt{(-\beta_2)}}\int_{-\infty}^{\infty}\omega^4e^{-\omega^2}\mathrm{d}\omega+\right.$$

$$\left.+\frac{\alpha_0\beta_3{}^2}{-2\beta_2{}^3\sqrt{(-\beta_2)}}\int_{-\infty}^{\infty}\omega^6e^{-\omega^2}\mathrm{d}\omega\right\}\tau^{-\frac{3}{2}}+r_1(\tau), \tag{4.66}$$

where

$$\lim_{\tau\to\infty} \tau^{\frac{3}{2}} r_1(\tau) = 0.$$

In fact,

$$r_1(\tau) = r(\tau) - \left\{ \frac{\alpha_2}{-\beta_2\sqrt{(-\beta_2)}} \left[\int\limits_{-\infty}^{-\sqrt{(-\beta_2\tau)}\epsilon} \omega^2 e^{-\omega^2} \mathrm{d}\omega + \int\limits_{\sqrt{(-\beta_2\tau)}\epsilon}^{\infty} \omega^2 e^{-\omega^2} \mathrm{d}\omega \right] + \right.$$

$$+ \frac{\alpha_1\beta_3 + \alpha_0\beta_4}{\beta_2^2\sqrt{(-\beta_2)}} \left[\int\limits_{-\infty}^{-\sqrt{(-\beta_2\tau)}\epsilon} \omega^4 e^{-\omega^2} \mathrm{d}\omega + \int\limits_{\sqrt{(-\beta_2\tau)}\epsilon}^{\infty} \omega^4 e^{-\omega^2} \mathrm{d}\omega \right] +$$

$$\left. + \frac{\alpha_0\beta_3^2}{-2\beta_2^3\sqrt{(-\beta_2)}} \left[\int\limits_{-\infty}^{-\sqrt{(-\beta_2\tau)}\epsilon} \omega^6 e^{-\omega^2} \mathrm{d}\omega + \int\limits_{\sqrt{(-\beta_2\tau)}\epsilon}^{\infty} \omega^6 e^{-\omega^2} \mathrm{d}\omega \right] \right\} \tau^{-\frac{3}{2}} -$$

$$- \left\{ \frac{\alpha_0}{\sqrt{(-\beta_2)}} \left[\int\limits_{-\infty}^{-\sqrt{(-\beta_2\tau)}\epsilon} e^{-\omega^2} \mathrm{d}\omega + \int\limits_{\sqrt{(-\beta_2\tau)}\epsilon}^{\infty} e^{-\omega^2} \mathrm{d}\omega \right] \right\} \tau^{-\frac{1}{2}},$$

and as the first six integrals on the right hand side of this equation, obviously tend to zero as $\tau \to \infty$, it only remains to show, that

$$\lim_{\tau\to\infty} \tau \int\limits_{-\infty}^{-\sqrt{(-\beta_2\tau)}\epsilon} e^{-\omega^2} \mathrm{d}\omega = \lim_{\tau\to\infty} \tau \int\limits_{\sqrt{(-\beta_2\tau)}\epsilon}^{\infty} e^{-\omega^2} \mathrm{d}\omega = 0;$$

however the correctness of these relations follows, for example, from the fact that

$$\int\limits_{-\infty}^{-\sqrt{(-\beta_2\tau)}\epsilon} e^{-\omega^2} \mathrm{d}\omega < e^{\frac{1}{2}\beta_2\tau\epsilon^2} \int\limits_{-\infty}^{-\sqrt{(-\beta_2\tau)}\epsilon} e^{-\frac{1}{2}\omega^2} \mathrm{d}\omega,$$

and similarly for the other integrals.

Now, taking account of the fact that

$$\int_{-\infty}^{\infty} e^{-\omega^2}\mathrm{d}\omega = \sqrt{\pi}, \qquad \int_{-\infty}^{\infty} \omega^2 e^{-\omega^2}\mathrm{d}\omega = \tfrac{1}{2}\sqrt{\pi},$$

$$\int_{-\infty}^{\infty} \omega^4 e^{-\omega^2}\mathrm{d}\omega = \frac{3}{4}\sqrt{\pi}, \qquad \int_{-\infty}^{\infty} \omega^6 e^{-\omega^2}\mathrm{d}\omega = \frac{15}{8}\sqrt{\pi},$$

we obtain from (4.66) the statement of the lemma. It is not difficult to see, that the same reasoning, continued farther in a similar way, can give any number of terms of the asymptotic expansion of the integral considered, and also an estimate of the remainder term. For this it is only necessary to continue to higher powers of $\tau^{-\frac{1}{2}}$ the expansion (4.65).

Example 1. *The asymptotic expansion of the gamma function.* We shall search for the asymptotic expansion of $(\tau > 0)$

$$\Gamma(\tau+1) = \int_{0}^{\infty} x^{\tau}e^{-x}\mathrm{d}x.$$

Putting $x = \tau(\lambda+1)$, we find, that

$$\Gamma(\tau+1) = \tau^{\tau+1}e^{-\tau}\int_{-1}^{\infty} \{(\lambda+1)e^{-\lambda}\}^{\tau}\mathrm{d}\lambda =$$

$$= \tau^{\tau+1}e^{-\tau}\int_{-1}^{\infty} e^{\tau\{\ln(1+\lambda)-\lambda\}}\mathrm{d}\lambda.$$

To this integral the preceding lemma is applicable, in which it is necessary to put $\alpha(\lambda) = 0$ for $\lambda < -1$, $\alpha(\lambda) = 1$ for $\lambda > -1$, $\beta(\lambda) = 0$ for $\lambda < -1$ and $\beta(\lambda) = \ln(1+\lambda)$ for $\lambda > -1$. Here, consequently, $\alpha_0 = 1$, $\alpha_1 = 0$, $\alpha_2 = 0$, $\beta(\lambda) = -\lambda^2/2+\lambda^3/3-\lambda^4/4+\ldots$ that is, $\beta_2 = -\frac{1}{2}$, $\beta_3 = \frac{1}{3}$, $\beta_4 = -\frac{1}{4}$. Therefore, by (4.63)

$$\int_{-1}^{\infty} e^{\tau\{\ln(1+\lambda)-\lambda\}}\mathrm{d}\lambda \sim \sqrt{(2\pi)}\tau^{-\frac{1}{2}}+\tfrac{1}{12}\sqrt{(2\pi)}\tau^{-\frac{3}{2}}+\ \ldots,$$

and this indicates, that

$$\lim_{\tau\to\infty} \frac{e^\tau \Gamma(\tau+1)}{\tau^{\tau+\frac{1}{2}}} = \sqrt{(2\pi)},$$

and also, that

$$\lim_{\tau\to\infty} \tau \left\{ \frac{e^\tau \Gamma(\tau+1)}{\sqrt{(2\pi)}\tau^{\tau+\frac{1}{2}}} - 1 \right\} = \frac{1}{12}.$$

The first of these limiting relations bears the title of Stirling's formula. In the simple case considered a greater number of terms of the asymptotic expansion can also easily be obtained.

Stirling's formula can be written in the following form:

$$\lim_{\tau\to\infty} \sqrt{\tau} \int_0^\infty (xe^{1-x})^\tau \mathrm{d}x = \sqrt{(2\pi)}.$$

This equation is equivalent to the first of the limiting relations used in example 1, Art. 23 (page 210)

$$\lim_{\tau\to\infty} \sqrt{\tau} \int_0^\infty \frac{\mu}{x+\mu} (xe^{1-x})^\tau \mathrm{d}x = \frac{\mu}{1+\mu} \sqrt{(2\pi)},$$

which is a consequence of the relation (4.64b). In fact,

$$\int_0^\infty \frac{\mu}{x+\mu} (xe^{1-x})^\tau \mathrm{d}x = \int_{-1}^\infty \frac{\mu}{1+\lambda+\mu} \{1+\lambda)e^{-\lambda}\}^\tau \mathrm{d}\lambda =$$

$$= \int_{-1}^\infty \frac{\mu}{1+\lambda+\mu} e^{\tau\{\ln(1+\lambda)-\lambda\}} \mathrm{d}\lambda.$$

In this integral

$$\alpha(\lambda) = \frac{\mu}{1+\lambda+\mu} (\lambda > -1),$$

so that $\alpha_0 = \mu/(1+\mu)$, and $\beta(\lambda) = \ln(1+\lambda) - \lambda$ so that $\beta_2 = -\frac{1}{2}$.

Let us now consider

$$\int_L a(p)e^{\tau b(p)}\mathrm{d}p,$$

where $a(p)$ and $b(p)$ are functions, regular in a certain domain, and the path L lies within this domain. It is obvious, that for larger values of τ the magnitude of this integral is mainly determined by the section of the path L, on which $\operatorname{Re} b(p)$ is large in comparison with its remaining values on L, and that this integral is the easier to evaluate, the smaller this section is and the more quickly $\operatorname{Re} b(p)$ decreases outside this section. In connexion with this it appears to be useful to deform, if this is possible, the path L in such a way that on it there is a point, at which $\operatorname{Re} b(p)$ has a greater value, than all the others, which it assumes on L.

However, in order for it to be possible to apply the preceding lemma, we must also arrange that along the path $\operatorname{Im} b(p)$ will have a constant value. On such a path $\operatorname{Re} b(p)$ can attain a maximum only at the point, at which $b'(p) = 0$. In fact, if the length of the arc of the path, measured from the point of maximum $\operatorname{Re} b(p)$ is denoted by λ and on this path we put $b(p) = \beta(\lambda)+i\beta_1$, where $\beta(\lambda)$ is a real function, and $\beta_1 = \operatorname{Im} b(p) = \text{const.}$ then $\beta'(0) = 0$ and, consequently, $\partial b/\partial\lambda|_{\lambda-0} = 0$, that is in general at this point $b'(p) = 0$. Such a point is, as is well known, a saddle point of the surface $\operatorname{Re} b(p)$ as a function of two variables (the real and imaginary parts of p), and the path goes "across the pass", that is along the projection of the principal section of this surface, which has a maximum at the saddle point (Fig. 24).

Therefore, let us assume, that along the path L, $\operatorname{Im} b(p) = \beta_1 = \text{const.}$ and $\operatorname{Re} b(p) = \beta(\lambda)(-\epsilon \leqslant \lambda \leqslant \epsilon)$ and that on the remaining sections of the path $\operatorname{Re} b(p) < \beta_0-\delta$, where $\beta_0 = \beta(0)$ is the greatest value of $\operatorname{Re} b(p)$ on L and $\delta > 0$. Then, as is easily seen (see the beginning of the proof of the lemma), the asymptotic expansion

$$\int_L a(p)e^{\tau b(p)}\mathrm{d}p$$

is identical with the asymptotic expansion

$$\int_{-\epsilon}^{\epsilon} \alpha(\lambda)e^{\tau\{\beta(\lambda)+i\beta_1\}}\mathrm{d}\lambda, \tag{4.67}$$

where $\alpha(\lambda) = a\{p(\lambda)\}(\mathrm{d}p/\mathrm{d}\lambda)$, and $p = p(\lambda)$ is the equation of the section $-\epsilon \leqslant \lambda \leqslant \epsilon$ of the path L.

Thus, from the lemma of the present article there follows the following

THEOREM 12. *Let $a(p)$ and $b(p)$ be regular in a certain domain, which contains the point p_0, at which $b'(p_0) = 0$, and let L be a path, which lies in this domain and passes through the point p_0 in such a way that on a certain segment of this path, which contains the point p_0,*

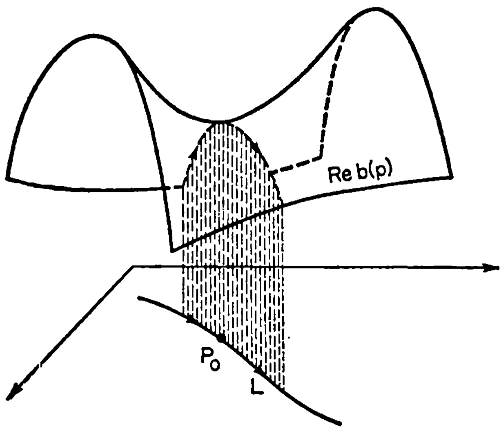


FIG. 24

$\operatorname{Im} b(p) = \beta_1 = const.$ *Also, let* $\operatorname{Re} b(p) < \operatorname{Re} b(p_0) - \delta$, $(\delta > 0)$, *everywhere on L outside this segment. Then as $\tau \to \infty$*

$$e^{-\tau b(p_0)} \int_L a(p) e^{\tau b(p)} \mathrm{d}p \sim \sqrt{\left(\frac{2\pi}{|b''(p_0)|}\right)} a(p_0) e^{i\theta} \tau^{-\frac{1}{2}}, \tag{4.68}$$

if the integral converges for all sufficiently large τ. In other words,

$$\lim_{\tau \to \infty} \tau^{\frac{1}{2}} \left\{ e^{-\tau b(p_0)} \int_L a(p) e^{\tau b(p)} \mathrm{d}p - \right.$$

$$\left. - \sqrt{\left(\frac{2\pi}{|b''(p_0)|}\right)} a(p_0) e^{i\theta} \tau^{-\frac{1}{2}} \right\} = 0,$$

or

$$\lim_{\tau\to\infty} \tau^{\frac{1}{2}} e^{-\tau b(p_0)} \int_L a(p) e^{\tau b(p)} \mathrm{d}p = \sqrt{\left(\frac{2\pi}{|b''(p_0)|}\right)} a(p_0) e^{i\theta},$$

where $e^{i\theta} = (\mathrm{d}p/\mathrm{d}\lambda)_{p=p_0}$, *and* λ *is the length of the arc of the path* L, *measured from the point* p_0 *in the direction of integration.*

In fact, for the integral (4.67) from the expansion (4.63) the relation (4.68) follows (where we calculate only the first term of the expansion) if it is considered, that in the given case $\beta_2 = -\frac{1}{2}|b''(p_0)|$ and $\alpha_0 = a(p_0)(\mathrm{d}p/\mathrm{d}\lambda)_{p=p_0}$.

The expansion (4.68) can, of course, be continued by using the second term of the expansion (4.63), for which it is necessary to put

$$\beta_3 = \frac{1}{6}\frac{\mathrm{d}^3}{\mathrm{d}\lambda^3}\mathrm{Re}\,b(p)\bigg|_{\lambda=0}, \quad \beta_4 = \frac{1}{24}\frac{\mathrm{d}^4}{\mathrm{d}\lambda^4}\mathrm{Re}\,b(p)\bigg|_{\lambda=0}$$

and calculate α_1 and α_2 from the expansion

$$a\{p(\lambda)\}\frac{\mathrm{d}p}{\mathrm{d}\lambda} = \alpha_0 + \alpha_1\lambda + \alpha_2\lambda^2 + \dots$$

Example 2. Let us consider the integral

$$\frac{1}{2\pi i}\int_L h(p) e^{\tau\sqrt{[(p-p_0)^2+\mu^2]}} e^{pt} \mathrm{d}p,$$

where the path L is the straight line $\mathrm{Re}\,p = \mathrm{Re}\,p_0$, μ is a real number, different from zero, and $h(p)$ is a function regular on L such that the integral converges. Then $b(p) = \sqrt{[(p-p_0)^2+\mu^2]}$, $b'(p_0) = 0$ and on L, $p = \mathrm{Re}\,p_0 + i(\mathrm{Im}\,p_0 + \lambda)$, that is $b(p) = \sqrt{(\mu^2-\lambda^2)}$, and consequently $\mathrm{Im}\,b(p) = 0$ for $-|\mu| \leqslant \lambda \leqslant |\mu|$, that is the path L satisfies the conditions of theorem 12. In addition to this, outside the segment $-|\mu| \leqslant \lambda \leqslant |\mu|$ of the path L we have

$$\mathrm{Re}\,b(p) = 0 < \mathrm{Re}\,b(p_0) = |\mu|$$

so that all the conditions of theorem 12 are satisfied. Consequently, taking into account, that $b''(p_0) = |\mu|^{-1}$ and $\theta = \pi/2$, if the path L is oriented in the direction of increase of $\mathrm{Im}\,p$, we find from the

expansion (4.68), that

$$e^{-\tau|\mu|}\frac{1}{2\pi i}\int\limits_L h(p)e^{\tau\sqrt{[(p-p_0)^2+\mu]}}e^{pt}\mathrm{d}p \sim \sqrt{\left(\frac{|\mu|}{2\pi}\right)}h(p_0)e^{p_0t}\tau^{-\frac{1}{2}}+\ \ldots$$

Integrals of this kind are encountered in certain problems of mathematical physics, in particlar, in the problem of the propagation of electro magnetic waves along lines and generally in the solution of the one dimensional wave equation with a dispersion term.†

† See, for example, V. I. Levin and U. I. Grosberg, *The Differential Equations of Mathematical Physics*, Gostekhizdat, 1951, Chapter 1, and also A. M. Efros and A. M. Danilevskii, *The Operational Calculus and Contour Integrals*, ONTI, 1937, pages 213–216.

CHAPTER V

HURWITZ'S PROBLEM FOR POLYNOMIALS

In the present chapter results are considered which relate to the distribution of the zeros of polynomials in the complex plane and which play an important part in applications, mainly in questions of stability. The central place in this chapter is occupied by Hurwitz's problem, which is formulated and solved. From its solution and in part independently of it, certain criteria are obtained on the location of the roots of a polynomial in a definite region of the complex plane, which are suitable for practical applications, among those referred to here being the results of I. A. Vishnegradskii and other authors.

26. Statement of the problem. Examples

In the integration of ordinary linear differential equations with constant coefficients the roots of the characteristic polynomial (the denominator $Q(p)$ of the rational Laplace transform of the solution of this equation, see example 5, Art. 19) have to be found. If q_n are the roots of the polynomial $Q(p)$, the general solution of the homogeneous linear differential equation with constant coefficients will have the form (see example 5, Art. 21, formula (3.61)):

$$y_0(t) = \sum_{n=1}^{N} M_n(t) e^{q_n t}, \tag{5.1}$$

where the $M_n(t)$ are polynomials of degree $m_n - 1$ with arbitrary constant coefficients, m_n is the multiplicity of the root q_n and N is the number of different roots of the polynomial (Qp) (hence, $m_1 + \ldots + m_N$ is its degree, that is the order of the equation).

In many problems the behaviour of the function $y_0(t)$ as $t \to \infty$, that is for sufficiently great values of t is extremely important. It is obvious that it is determined (as is, by the way, the behaviour of the solution for all t) exclusively by the roots q_n of the polynomial $Q(p)$ and the coefficients of the polynomial $M_n(t)$, which depend on

the initial conditions. Let us consider the individual terms $M_n(t)e^{q_n t}$ of the sum (5.1). Three cases are possible: $\operatorname{Re} q_n < 0$, $\operatorname{Re} q_n = 0$ and $\operatorname{Re} q_n > 0$. In the first case, if $q_n = -\lambda+i\omega$, where $\lambda > 0$, then

$$\lim_{t\to\infty} M_n(t)e^{q_n t} = 0,$$

that is, if we also suppose, that q_n is a simple root ($m_n = 1$), and consequently, $M_n(t) = \text{const.}$, the expression const. $e^{-\lambda t}\,e^{i\omega t}$ represents a *damped* harmonic oscillation. In the second case $q_n = i\omega$, if this root is simple, we obtain the expression const. $e^{i\omega t}$, which is an *undamped* harmonic oscillation of frequency ω. Finally, in the third case $q_n = \lambda+i\omega_1\,\lambda > 0$ and $M_n(t)e^{\lambda t}\,e^{i\omega t}$ represents an alternating oscillation, the amplitude of which assumes arbitrarily great values with increase of the time.†

As we shall see, these three qualitatively quite different cases occur depending on the sign of the real part of the root q_n, while boundedness of the solution as $t \to \infty$ can only take place, when *all the roots of the polynomial have non-positive real parts*‡ and the solution can only tend to zero as $t \to \infty$ when *all the roots q_n have negative real parts.*

The signs of the real parts of the roots of polynomials play an important part also in other similar questions of an applied character, but what has already been said will be sufficient to show the importance of the statement and solution of the following problem (called *Hurwitz's problem*).

Let us be given the polynomial

$$f(z) = a_0 z^n + a_1 z^{n-1} + \ldots + a_n.$$

If all the roots of this polynomial lie in the left hand half-plane $\operatorname{Re} z < 0$, we shall call $f(z)$ an *H-polynomial* or say, that *$f(z)$ belongs to the class (H)*. If, in addition, purely imaginary roots are present $f(z)$ will be called an H'-polynomial or it will be said, that *$f(z)$ belongs to the class (H')*. *What are the necessary and sufficient conditions, which must be satisfied by the coefficients of a polynomial in order that the polynomial should belong to the class (H) or (H')?*

† If in this case $\omega = 0$, we do not have an oscillation, but $M_n(t)\exp(\lambda t)$ increases without limit as $t\to\infty$.

‡ This is not sufficient for boundedness of the solution: it is also necessary, that the polynomials $M_n(t)$, which correspond to the purely imaginary roots, should reduce to constants.

It must also be possible to apply these conditions in practice, that is, they must be such that the testing of their satisfaction for concrete polynomials does not present too much difficulty. Conditions, which are only sufficient will also have practical value, if they are particularly simple.

Some other questions which are important for applications arise in connexion with this basic problem. Let us pause at one of them, which will be analysed in greater detail below (see Art. 29). In almost all applied problems the coefficients of the polynomial $f(z)$ depend on certain parameters. For example, in the coefficients of the characteristic polynomial of the differential equation of motion of a linear oscillating system mz^2+hz+c there are present as parameters: m the mass of the oscillating point, h the coefficient of resistance (it is assumed that the resistance is proportional to the velocity) and c the coefficient of elasticity (the stiffness of the spring), in the case of the equation of an RLC-circuit these parameters have other physical meanings and so on. It is clear, that the roots of a polynomial the coefficients of which depend on parameters, will also be functions of these parameters. It is thus possible, that for some sets of values of these parameters the polynomial considered will belong to the class (H) or (H'), and for other sets of values this will not be so. Thus, the problem arises of *the selection of those sets of values of the parameters, for which the polynomial considered is an H- or an H'-polynomial.*

If the coefficients depend only on one parameter ξ, then as a rule, the values of ξ which interest us will occupy one or more intervals of the ξ-axis; if the parameters depend on two parameters ξ and η, then in the ξ, η-plane the points which correspond to H- or H'-polynomials, will occupy one or several domains and so on. These domains on the ξ-axis, in the ξ, η-plane and so on, are called—in view of the basic physical content of the problems, which lead to the problem considered—*domains of stability.* The cases which are most important and have been the most investigated are those, in which the coefficients of the polynomial depend linearly on one or two parameters.

As a first acquaintance with the type of question described above, let us consider some simple examples.

Example 1. *A polynomial of the first degree.* The polynomial $a_0z+a_1(a_0 \neq 0)$ has the single root $-a_1/a_0$ and, hence, the necessary

and sufficient condition that this polynomial should belong to the class (H) or (H') respectively, is contained in the inequality $\operatorname{Re} a_1/a_0 > 0$, or $\operatorname{Re} a_1/a_0 \geqslant 0$ respectively.

Example 2. *A polynomial of the second degree with real coefficients.* The polynomial $a_0z^2+a_1z+a_2 (a_0 \neq 0)$, where a_0, a_1 and a_2 are real numbers, has roots $[-a_0 \pm \sqrt{(a_1^2-4a_0a_2)}]/2a_0$. It is not difficult to see that the necessary and sufficient conditions for this polynomial to belong to the class (H), are the inequalities $a_0a_1 > 0$, $a_0a_2 > 0$. These conditions, in other words, amount to the requirement that all three coefficients must have one sign. In fact, if $a_1 = 0$, the roots will be imaginary, or real with different signs, or equal to zero; if $a_1 = 0$, one root will be equal to zero: if $a_0a_1 < 0$, that is $a_1/a_0 < 0$, then at least one of the roots will have a positive real part; if, finally, $a_0a_2 < 0$, then one root is positive, and the other negative. If however all the coefficients are of one sign, it is easy to see, that the polynomial of the second degree will belong to the class (H).† Also, for the polynomial to belong to the class (H'), it is necessary and sufficient, that one of the following sets of conditions should be satisfied: $a_0 \neq 0$ and $a_0a_1 \geqslant 0$, $a_0a_2 \geqslant 0$, or $a_0 = 0$ and $a_1a_2 > 0$, or $a_0 = a_2 = 0$ and $a_1 \neq 0$. In fact, if $a_1 = 0$ and $a_2 = 0$ but $a_0 \neq 0$, both roots will be equal to zero; if $a_2 = 0$ and $a_0a_1 > 0$, one root will be equal to zero, and the other negative; if $a_1 = 0$ and $a_0a_2 > 0$, both roots will be imaginary; also, if $a_0 = 0$ (this means that one root has become infinite; the corresponding root will be found on the boundary of the left hand half-plane, as the point at infinity will belong to the imaginary axis) and $a_1a_2 > 0$, then the unique positive root will be negative; finally, if $a_2 = a_0 = 0$ and $a_1 \neq 0$, the finite root will be equal to zero.

Let us now suppose, that the coefficients a_0, a_1 and a_2 depend linearly on two parameters ξ and η: $a_0 = a'_0+a''_0\,\xi+a'''_0\eta$, $a_1 = a'_1 + a''_1\xi + a'''_1\eta$, $a_2 = a'_2 + a''_2\xi + a'''_2\eta$. Then the condition for the polynomial to belong to the class (H) is expressed by the

† It must be noted, that any polynomial with real coefficients cannot belong to the class (H), if its coefficients do not have the same sign, that is, in the case of real coefficients the identity of their signs is a necessary condition for the polynomial to belong to the class (H). This follows from the fact that a polynomial with real coefficients can be decomposed into real linear and quadratic factors, which if the polynomial belongs to the class (H) will necessarily have coefficients of the same sign (for linear factors this is obvious, for quadratic factors we have shown this).

inequalities

$$(a_0'+a_0''\xi+a_0'''\eta)(a_1'+a_1''\xi+a_1'''\eta) > 0, \tag{5.2}$$

$$(a_0'+a_0''\xi+a_0'''\eta)(a_2'+a_2''\xi+a_2'''\eta) > 0. \tag{5.3}$$

Let us find their geometrical meaning. The equation

$$a_2'+a_2''\xi+a_2'''\eta = 0$$

is the equation of a straight line in the (ξ, η)-plane, and the inequalities $a'_2+a_2''\xi+a_2'''\eta > 0$ and $a'_2+a_2''\xi+a_2'''\eta < 0$ characterize each one of the half-planes, into which this straight line divides the plane.

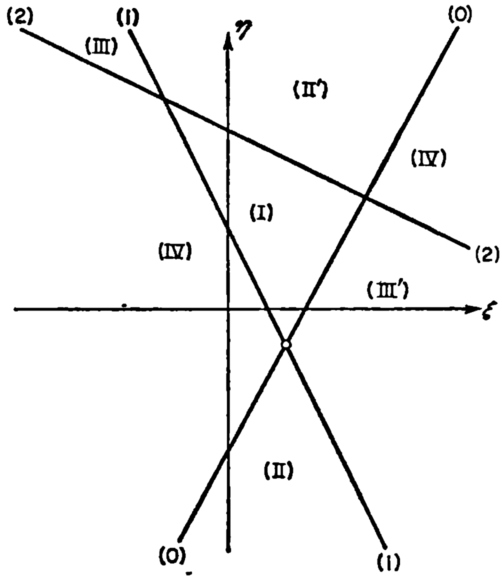


FIG. 25

Let us draw the straight lines *(1)* $a_1'+a_1''\xi+a_1'''\eta = 0$ and *(0)* $a_0'+a_0''\xi+a_0'''\eta = 0$ (Fig. 25).† Then the inequality (5.2), obviously, indicates that the point (ξ, η) must lie in one of the two pairs of angular sectors into which the straight lines *(0)* and *(1)* divide the plane. Let us now draw the straight line *(2)* $a_2'+a_2''\xi+a_2'''\eta = 0$ and suppose initially that it does not pass through the point of intersection of the straight lines *(0)* and *(1)*. The inequality (5.3) indicates, that the point (ξ, η) must lie in one of the pair of angular

† We assume, that the straight lines *(0)* and *(1)* are not parallel, and also, that the straight line *(2)* is not parallel to either of them. The analysis of the remainder of the cases which are possible here is left to the reader.

sectors, into which the straight lines *(0)* and *(2)* divide the plane. The simultaneous satisfaction of conditions (5.2) and (5.3) indicates, then, that the point (ξ, η) must belong to a certain domain D, which can (depending on the numbers $a_0'a_0'', \ldots, a_2'''$) only be the triangle *(I)*, or consist of the parts *(II)* and *(II')*, which can be considered as an infinite triangle, or *(III)* and *(III')*, or *(IV)* and *(IV')*. If, however, the straight line *(2)* passes through the point of intersection of the straight lines *(0)* and *(1)*, the domain D must either be empty, or consist of one of the three pairs of angular sectors, into which the straight lines *(0)*, *(1)* and *(2)* divide the plane.

This discussion also shows, what happens in the case where the coefficients of the equation depend linearly only on the single parameter ξ. For this it is only necessary to put η = constant and consider how the straight line, parallel to the ξ-axis, intersects the domain D. It is obvious that this intersection, that is the domain of values of ξ, for which the polynomial will belong to the class D, will either be empty, or will consist of one finite, or of one semi-infinite, or of two semi-infinite intervals. Such are the domains of stability in these cases.

So far as the conditions for the polynomial to belong to the class (H') are concerned, by the above it is necessary for this, to add to the domain D its boundary except for the point of intersection of the straight lines *(0)* and *(1)*, if the straight line *(2)* does not pass through this point. In order to obtain these conditions in the case where the coefficients of the polynomial depend on a single parameter, the corresponding additions must be made to the intervals described above.

Let us consider two concrete examples.

(*a*) The polynomial $\xi+\eta+1+\xi z+\eta z^2$. Here the conditions for the polynomial to belong to the class (H) are the inequalities $\xi\eta > 0$ and $(\xi+\eta-1)\eta > 0$. Let us draw the straight lines *(0)*, $\eta = 0$, (1) $\xi = 0$ and *(2)* $\xi+\eta-1 = 0$ (Fig. 26a). The inequality $\xi\eta > 0$ indicates that the point (ξ, η) belongs to the first or third quadrant, and the condition $(\xi+\eta-1)\eta > 0$ that it belongs to the obtuse angular sector, formed by the straight lines *(0)* and *(2)*. Consequently, the polynomial considered will be an H-polynomial if and only if it is the case that the point (ξ, η) belongs to the shaded domain (not including its boundary).

(*b*) The polynomial $1-\xi-\eta+\xi\eta+\eta z^2$. The straight lines *(0)*, *(1)*, and *(2)* are the same, but the condition $(1-\xi-\eta)\eta > 0$ now indicates

that the point (ξ, η) belongs to the acute angular sectors, formed by the straight lines (*0*) and (*2*). Thus, this polynomial will be an H-polynomial only in the case where the point (ξ, η) lies in the shaded triangle (Fig. 26b).

In order that the polynomials considered in these examples should belong to the class (H'), it is necessary and sufficient, that the point (ξ, η) should belong to the shaded regions including their boundaries, but excluding the co-ordinate origin (the point of intersection of the straight lines (*0*) and (*1*), as in general this point does not correspond to such an equation.

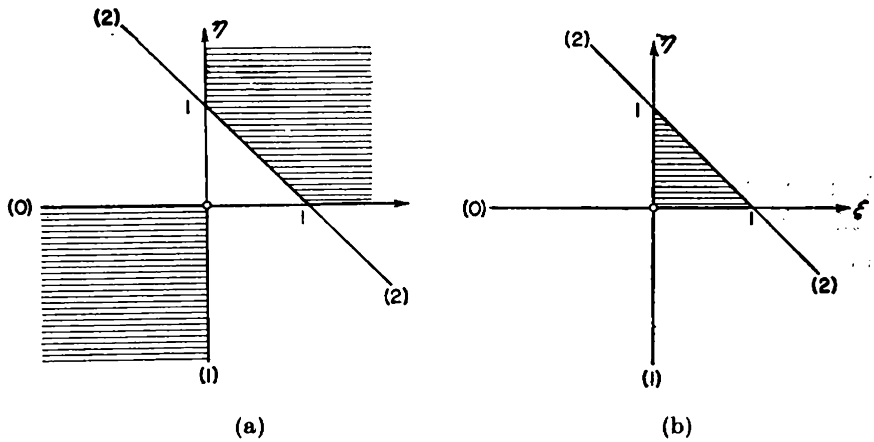


FIG. 26

27. Other forms of statement of the problem. The simplest criteria

In some cases it is useful to put the problem considered into a somewhat different form by a transformation of the independent complex variable. If we put $z = \alpha w + \beta$, where α and β are arbitrary complex coefficients, then $f(z) = a_0 z_n + a_1 z^{n-1} + \ldots + a_n$ transforms into $f(\alpha w + \beta) = g(w)$, where $g(w)$, is obviously also a polynomial of degree n. It is easy to see that if $f(z)$ is an H-polynomial, all the roots of $g(w)$ will belong to the half-plane $\operatorname{Re}(\alpha w + \beta) < 0$, and conversely, if $g(w)$ is a polynomial all the roots of which belong to the half-plane $\operatorname{Re}(\alpha w + \beta) < 0$, then $f(z)$ will belong to the class (H). In particular, if $z = iw$, then $g(w) = f(iw)$ will have all its roots in the upper half-plane $\operatorname{Im} w > 0$, if $f(z)$ belongs to the class (H), and conversely.

We shall obtain a similar result if we apply not a linear, but a bilinear transformation. Let $z = (\alpha w+\beta)/(\gamma w+\delta)$, where α, β, γ, δ are arbitrary complex coefficients, $\alpha\delta-\beta\gamma \neq 0$. Then, as is not difficult to see, $f(z) = f[(\alpha w+\beta)/(\gamma w+\delta)]$ is represented in the form of the rational function $g(w)/(\gamma w+\delta)^n$, where $g(w) = b_0w^n+b_1w^{n-1}+ + \ldots +b_n$ is a polynomial of degree n. If $f(z)$ belongs to the class (H), then all the roots of $g(w)$ will lie inside or outside a certain curve, or in a certain half-plane, namely in that domain, into which the bilinear transformation $z = (\alpha w+\beta)/(\gamma w+\delta)$ maps the half-plane $\operatorname{Re} z < 0$,† and conversely, if all the roots of $g(w)$ lie in this domain, then $f(z)$ is an H-polynomial. In particular, if $z = (\alpha w+\bar{\alpha})/(w-1)$, where $\operatorname{Re}\alpha > 0$, then the image of the half-plane $\operatorname{Re} z < 0$ is the unit circle $|w| < 1$ of the w-plane. There exist easily verifiable sufficient conditions for all the roots of a polynomial to belong to the unit circle with centre at the origin, that is conditions sufficient for all the roots of a polynomial to have a modulus less than unity. Each of these conditions, obviously gives conditions which are sufficient for $f(z)$ to be an H-polynomial. Let us prove one of the simplest conditions of this type.

THEOREM 1. *If*

$$g(w) = (w-1)^n f\left(\frac{\alpha w+\bar{\alpha}}{w-1}\right) = b_0w^n+b_1w^{n-1}+ \ldots +b_n$$

and there exists a number α with a positive real part, such that

$$|b_1|+|b_2|+ \ldots +|b_n| < |b_0|, \tag{5.4}$$

then $f(z)$ will belong to the class (H).

By the above it is sufficient to prove, that if the inequality (5.4) is satisfied, all the roots of the polynomial $g(w)$ will lie inside the circle $|w| < 1$. Let us assume, that $g_0(w_0) = 0$ and $|w_0| \geqslant 1$. Then

$$b_0w_0^n+b_1w_0^{n-1}+ \ldots +b_{n-1}w_0+b_n = 0$$

and, consequently,

$$b_0 = -\frac{b_1}{w_0} - \ldots - \frac{b_{n-1}}{w_0^{n-1}} - \frac{b_n}{w_0^n},$$

and hence

$$|b_0| \leqslant \frac{|b_1|}{|w_0|} + \ldots + \frac{|b_{n-1}|}{|w_0|^{n-1}} + \frac{|b_n|}{|w_0|^n} \leqslant |b_1|+ \ldots +|b_{n-1}|+|b_n|.$$

† See F.C.V., Chap. II, Art 18.

Thus, the inequality (5.4) sets the bound $|w_0| < 1$ for any root w_0 of the polynomial $g(w)$, and the theorem is proved.

Example 3. The investigation of the polynomial $z^3+2z^2+2z+\xi$. In order to establish, for what values of $\xi > 0$† this polynomial will belong to the class (H), let us apply theorem 1. For this purpose let us form $g(w)$ for real $\alpha > 0$:

$$g(w) = (w-1)^3 \left\{ \alpha^3\left(\frac{w+1}{w-1}\right)^3 + 2\alpha^2\left(\frac{w+1}{w-1}\right)^2 + 2\alpha\left(\frac{w+1}{w-1}\right) + \xi \right\} =$$

$$= b_0 w^3 + b_1 w^2 + b_2 w + b_3,$$

where

$$\begin{aligned} b_0 &= \alpha^3+2\alpha^2+2\alpha+\xi, \\ b_1 &= 3\alpha^3+2\alpha^2-2\alpha-3\xi, \\ b_2 &= 3\alpha^3-2\alpha^2-2\alpha+3\xi, \\ b_3 &= \alpha^3-2\alpha^2+2\alpha-\xi. \end{aligned}$$

Consequently, the polynomial $z^3+2z^2+2z+\xi$ will in any case be an H-polynomial for every $\xi > 0$, for which it is possible to find an $\alpha > 0$, such that‡

$$\alpha^3+2\alpha^2+2\alpha+\xi > |\alpha^3-2\alpha^2+2\alpha-\xi| + \\ + |3\alpha^3-2\alpha^2-2\alpha+3\xi| + |3\alpha^3+2\alpha^2-2\alpha-3\xi|. \tag{5.5}$$

Let us suppose, that ξ and α are such that $b_1 \geqslant 0, b_0 \geqslant 0, b_3 \geqslant 0$, that is, such that

$$\xi \leqslant \alpha^3+\tfrac{2}{3}\alpha^2-\tfrac{2}{3}\alpha, \tag{5.6}$$

$$\xi \geqslant -\alpha^3+\tfrac{2}{3}\alpha^2-\tfrac{2}{3}\alpha, \tag{5.7}$$

$$\xi \leqslant \alpha^3-2\alpha^2+2\alpha. \tag{5.8}$$

Then from the inequality (5.5) we find, that

$$\alpha^3-2\alpha^2+2\alpha+\xi > 7\alpha^3-2\alpha^2-2\alpha-\xi,$$

that is

$$\xi > 3\alpha^3-2\alpha^2-2\alpha. \tag{5.9}$$

† By the remark in the footnote on page 248 for $\xi < 0$ the polynomial considered automatically cannot be an H-polynomial, and for $\xi = 0$ it has the root zero.

‡ Let us note, that with the given assumptions b_0 is always positive.

Hence, in this case α must be such, that $3\alpha^3-2\alpha^2-2\alpha > 0$ that is, $\alpha > \alpha_0 = 3(1+\sqrt{7})$. But then the inequality (5.7) is automatically satisfied, and ξ has only to satisfy the inequalities (5.6), (5.8) and (5.9) for $\alpha > \alpha_0$. But these inequalities are common to all α, less than α_1, where α_1 is the least of the positive abscissas of the points of intersection of the parabolas $\beta = \alpha^3-2\alpha^2+2\alpha$ and $\beta = \alpha^3+ +\frac{2}{3}\alpha^2-\frac{2}{3}\alpha$ with the parabola $\beta = 3\alpha^3-2\alpha^2-2\alpha$. A simple calculation shows, that $\alpha_1 = \sqrt{2}$. For this value of α it follows from equation (5.8) that $\xi \leqslant 4(\sqrt{(2)}-1)$.

Thus, our discussion has shown that the polynomial $z^3+2z^2+2z+\xi$ belongs to the class (H), if $0 < \xi < 4(\sqrt{(2)}-1)$. It is possible to show by other methods (see example 1, Art. 29), that this polynomial belongs to the class (H) for $0 < \xi < 4$ and only for these values of ξ.

In forming the polynomial $g(w)$ by means of the bilinear transformation $x = (\alpha w+\bar{\alpha})/(w-1)$ it is possible also to make use of the arbitrariness in the choice of α in order to obtain some simplification in $g(w)$. Let us illustrate this idea by the example of a polynomial of the second degree with complex coefficients.

Example 4. *A polynomial of the second degree with complex coefficients.* Let $f(z) = a_0z^2+a_1z+a_2$, where a_1 and a_2 are complex numbers and a_0 is a real number (this can always be arranged by multiplying the polynomial by $e^{i\phi}$, which does not affect the roots of the polynomial). Then

$$g(w) = (w-1)^2 f\left(\frac{\alpha w+\bar{\alpha}}{w-1}\right) =$$

$$= a_2-a_1\bar{\alpha}+a_0\bar{\alpha}^2+\{-2a_2-a_1(\alpha-\bar{\alpha})+2a_0\alpha\bar{\alpha}\}w+(a_2+a_1\alpha+a_0\alpha^2)w^2.$$

Let us now choose α to be such that the coefficient

$$b_1 = -2a_2-a_1(\alpha-\bar{\alpha})+2a_0\alpha\bar{\alpha}$$

of w in $g(w)$ reduces to zero. Calculating its conjugate from the equation

$$-2a_2-a_1(\alpha-\bar{\alpha})+2a_0\alpha\bar{\alpha} = 0$$

and taking account of the fact that $\bar{a}_0 = a_0$, we obtain

$$2(a_2-\bar{a}_2)+(a_1+\bar{a}_1)(\alpha-\bar{\alpha}) = 0.$$

Let us assume that $a_1+\bar{a}_1 \neq 0$. Then

$$\alpha-\bar{\alpha} = -2\frac{a_2-\bar{a}_2}{a_1+\bar{a}_1},$$

and consequently,

$$\alpha\bar{\alpha} = \frac{a_2-a_1\dfrac{a_2-\bar{a}_2}{a_1+\bar{a}_1}}{a_0} = \frac{a_2\bar{a}_1+a_1\bar{a}_2}{a_0(a_1+\bar{a}_1)}.$$

Let us also assume that $a_1+\bar{a}_1$ and $a_2\bar{a}_1+a_1\bar{a}_2$ have the same sign, that is, that

$$(a_1+\bar{a}_1)(a_2\bar{a}_1+a_1\bar{a}_2) > 0. \tag{5.10}$$

Then there exists such an α, which reduces b_1 to zero and for which $\operatorname{Re}\alpha > 0$ (since, it is always possible to find an α, with positive real part, and arbitrarily prescribed modulus and imaginary part). For this α the polynomial $g(w)$ assumes the form

$$a_0\bar{\alpha}^2-a_1\bar{\alpha}+a_2+(a_0\alpha^2+a_1\alpha+a_2)w^2,$$

and consequently, the moduli of its roots will be equal to

$$\frac{a_0\bar{\alpha}^2-a_1\bar{\alpha}+a_2}{a_0\alpha^2+a_1\alpha+a_2}.$$

The polynomial $f(z)$ will, therefore, belong to the class (H), if

$$\frac{(a_0\bar{\alpha}^2-a_1\bar{\alpha}+a_2)(a_0\alpha^2-\bar{a}_1\alpha+\bar{a}_2)}{(a_0\alpha^2+a_1\alpha+a_2)(a_0\bar{\alpha}^2+\bar{a}_1\bar{\alpha}+\bar{a}_2)} < 1. \tag{5.11}$$

This inequality can be put in the form

$$(\alpha+\bar{\alpha})\{a_2\bar{a}_1+a_1\bar{a}_2+a_0(a_2-\bar{a}_2)(\alpha-\bar{\alpha})+a_0(a_1+\bar{a}))\alpha\bar{\alpha}\} > 0,$$

and as $\alpha+\bar{\alpha} > 0$, it follows from this, that

$$a_2\bar{a}_1+a_1\bar{a}_2+a_0(a_2-\bar{a}_2)(\alpha-\bar{\alpha})+a_0(a_1+\bar{a}_1)\alpha\bar{\alpha} > 0, \tag{5.12}$$

or, substituting the calculated values of $\alpha-\bar{\alpha}$ and $\alpha\bar{\alpha}$,

$$a_2\bar{a}_1+a_1\bar{a}_2-a_0\frac{(a_2-\bar{a}_2)^2}{a_1+\bar{a}_1} > 0.$$

From this inequality and from the assumptions already made it follows, that $a_1+\bar{a}_1 > 0$, as if it is assumed, that $a_1+\bar{a}_1 < 0$, then

by (5.10) it would follow from this, that $a_2\bar{a}_1+a_1\bar{a}_2 < 0$ also, and this would entail (as $(a_2-\bar{a}_2)^2 \leqslant 0$)

$$a_2\bar{a}_1+a_1\bar{a}_2-a_0\frac{(a_2-\bar{a}_2)^2}{a_1+\bar{a}_1} < 0,$$

which contradicts what has been proved.

Therefore, the polynomial $f(z) = a_0z^2+a_1z+a_2$ automatically belongs to the class (H), if

$$a_0 > 0, \quad a_1+\bar{a}_1 > 0, \quad a_2\bar{a}_1+a_1\bar{a}_2 > 0. \tag{5.13}$$

We have proved up to the present, that the conditions (5.13) (the first of them is, of course, an assumption, which does not limit the generality) are sufficient for $f(z)$ to be an H-polynomial. In order to establish their necessity let us note, that the inequality (5.11) for any α with positive real part is a necessary condition for $f(z)$ to belong to the class (H), as the left hand side of this inequality is equal to the square of the modulus of the product of the roots of the polynomial $g(w)$. Consequently, the inequality (5.12) is also a necessary condition, which follows from the inequality (5.11). But from (5.12) for an α with a sufficiently small modulus it follows, that $a_2\bar{a}_1+a_1\bar{a}_2 > 0$; also, if $a_1+\bar{a}_1 = 0$, then in view of the arbitrariness of the sign and magnitude of $\alpha-\bar{\alpha}$ we must have $a_2-\bar{\alpha}_2 = 0$, and consequently,

$$a_2\bar{a}_1+a_1\bar{a}_2 = a_2(a_1+\bar{a}_1) = 0,$$

and the left hand side of the inequality (5.12) reduces to zero, which is inadmissible; therefore, $a_1+\bar{a}_1 \neq 0$; but then for an α with sufficiently large modulus it follows from (5.12), that $a_1+\bar{a}_1 > 0$, and the necessity of the last two conditions of (5.13) (on the assumption, that $a_0 > 0$) is proved. We shall see below, that this result also follows from the general criteria (for example, Hurwitz's criterion).

28. Hurwitz's criterion

We will now turn to the consideration of the general case. Let

$$f(z) = a_0z^n+a_1z^{n-1}+ \dots +\bar{a}_n = a_0\prod_{\nu=1}^{n}(z-z_\nu) \tag{5.14}$$

be the polynomial we want to investigate. Here the a_ν are arbitrary

complex numbers, where $a_0 \neq 0$, the z_ν are the roots of the polynomial. We want to find the conditions, which must be satisfied by the coefficients of the polynomial $f(z)$ in order that it should belong to the class of Hurwitz (H).

For this, following I. Schur, we shall consider the polynomial

$$f^*(z) = \bar{a}_0 \prod_{\nu=1}^{n} (z+\bar{z}_\nu). \tag{5.15}$$

Let us expand the given product. Using the expressions for the coefficients a_ν of the polynomial $f(z)$ in terms of its roots (which are used in the calculation of the product (5.14), we find, that

$$f^*(z) = \bar{a}_0 z^n - \bar{a}_1 z^{n-1} + \bar{a}_2 z^{n-2} - \dots + (-1)^n a_n. \tag{5.16}$$

It is obvious, that the roots z_ν of the polynomial (5.14) and the roots $-\bar{z}_\nu$ of the polynomial (5.16) are symmetrical with respect to one another relative to the imaginary axis. If the polynomial $f(z)$ belongs to the class (H), that is if all of its roots lie in the left hand half-plane, then the roots of the polynomial $f^*(z)$ occupy a symmetrical position in the right hand half-plane. Then for every point z of the right hand half-plane

$$|z - z_\nu| > |z + \bar{z}_\nu|,$$

and hence by virtue of the equations (5.14) and (5.15) it turns out, that

$$|f(z)| > |f^*(z)| \quad \text{for} \quad \mathrm{Re}\, z > 0. \tag{5.17}$$

Similarly we establish, that

$$|f(z)| = |f^*(z)| \quad \text{for} \quad \mathrm{Re}\, z = 0, \tag{5.18}$$

$$|f(z)| < |f^*(z)| \quad \text{for} \quad \mathrm{Re}\, z < 0. \tag{5.19}$$

It follows directly from this, that for the polynomial $f(z)$ to belong to the class (H) all the roots of the polynomial

$$g(z) = \alpha f(z) - \beta f^*(z), \quad \text{where} \quad |\alpha| > |\beta|, \tag{5.20}$$

must lie in the left hand half-plane, i.e. $g(z)$ also must belong to the class (H).

In fact, we have for $\operatorname{Re} z > 0$ and $\operatorname{Re} z = 0$ in virtue of the relations (5.17) and (5.18)

$$|\alpha f(z)| > |\beta f^*(z)|,$$

and hence also $g(z) \neq 0$.

It can be shown, that conversely from the fact that the polynomial $g(z)$ belongs to the class (H) it follows that the polynomial $f(z)$ also belongs to this class.

For this let us form (using the formula (5.16)) the polynomial $g^*(z)$. Then by virtue of the equation (5.20)

$$g^*(z) = \bar{\alpha} f^*(z) - \bar{\beta} f(z). \tag{5.21}$$

Eliminating the function $f^*(z)$ from equations (5.20) and (5.21), we find, that

$$f(z) = \frac{\alpha}{|\alpha|^2 - |\beta|^2} g(z) - \frac{\beta}{|\alpha|^2 - |\beta|^2} g^*(z). \tag{5.22}$$

Let us assume, that the polynomial $g(z)$ belongs to the class (H). Then, as we saw above, by virtue of the relation (5.22) the polynomial $f(z)$ will also belong to the given class; this proves our statement.

Let us now take the arbitrary fixed point ζ in the left hand half-plane (therefore, we have $\operatorname{Re} \zeta < 0$). Then, if the polynomial belongs to the class (H), it follows, as we have seen, that $|f(z)| < < |f^*(z)|$ and the polynomial

$$g(z) = f^*(\zeta) f(z) - f(\zeta) f^*(z) \tag{5.23}$$

also belongs to the class (H). It is obvious, that $g(\zeta) = 0$, and consequently, the polynomial $g(z)$ is divisible by $z - \zeta$; then the polynomial

$$f_1(z) = \frac{f^*(\zeta) f(z) - f(\zeta) f^*(z)}{z - \zeta} \tag{5.24}$$

will also be a member of the class (H).

Conversely, if for a certain point ζ of the left hand half-plane it is observed, that $|f(\zeta) < |f^*(\zeta)|$, and it happens that the polynomial $f_1(z)$, and with it also the polynomial $g(z)$, belongs to the class (H), then as we have seen, the polynomial $f(z)$ will also belong to this class. Therefore, we have proved

THEOREM 2. *Let ζ be a certain fixed number with negative real part. The polynomial of the n-th degree $f(z)$ will be a member of the Hurwitz*

class (H), *if and only if*, (1) $|f(\zeta)| < |f^*(\zeta)|$ and (2) *the polynomial of the* $(n-1)$*-th degree* $f_1(z)$ *defined by formula* (5.24), *belongs to the Hurwitz class* (H).

Let us now pass to the derivation of a proposition, which also reduces the question of whether the polynomial $f(z)$ of the n-th degree belongs to the class (H) to the question of whether some other polynomial of the $(n-1)$-th degree belongs to this class.

This polynomial in distinction from the polynomial (5.24) does not depend on the arbitrary parameter ζ.

Consider the polynomial

$$h(z) = \bar{a}_0 f(z) - a_0 f^*(z) \tag{5.25}$$

and let us again suppose, that the polynomial $f(z)$ belongs to the class (H). From the relations (5.17) and (5.19) it is obvious, that then all the roots of the polynomial $h(z)$ will lie on the imaginary axis. We shall show, that they are all simple roots of it. For this it is sufficient to establish, that the polynomials $h(z)$ and

$$h'(z) = \bar{a}_0 f'(z) - a_0 f^{*\prime}(z)$$

do not have common zeros on the imaginary axis. Let us assume the contrary. Let us assume, that at a certain point $z = iy_0$

$$h(iy_0) = h'(iy_0) = 0.$$

The last equation indicates, that the set of homogeneous equations

$$Xf(iy_0) + Yf^*(iy_0) = 0, \qquad Xf'(iy_0) + Yf^{*\prime}(iy_0) = 0 \tag{5.26}$$

can be satisfied by values $X = a_0$, $Y = \bar{a}_0$ different from zero. Then the determinant of the system (5.26) must be equal to zero. Thus, we arrive at the relation

$$\begin{vmatrix} f'(iy_0) & f^{*\prime}(iy_0) \\ f(iy_0) & f^*(iy_0) \end{vmatrix} = 0. \tag{5.27}$$

As $f(iy_0) \neq 0$ and $f^*(iy_0) \neq 0$, equation (5.27) can be rewritten in the form

$$\frac{f'(iy_0)}{f(iy_0)} = \frac{f^{*\prime}(iy_0)}{f^*(iy_0)},$$

or

$$[\ln f(z)]'_{z=iy_0} = [\ln f^*(z)]'_{z=iy_0}.$$

Using formulas (5.14) and (5.15), this equation can be put in the form

$$\sum_{\nu=1}^{n} \frac{1}{iy_0 - z_\nu} = \sum_{\nu=1}^{n} \frac{1}{iy_0 + z_\nu}.$$

However, the last relation cannot hold, as by our assumptions the real parts of all the terms on the right hand side are positive, and on the left hand side negative.†

Let us now establish that the degree of the polynomial $h(z)$ is equal to $n-1$. It is, in fact, obvious that it is less than n, and the coefficient of z^{n-1} is equal to

$$a_1 + \frac{a_0}{\bar{a}_0}\bar{a}_1 = \frac{a_1\bar{a}_0 + a_0\bar{a}_1}{\bar{a}_0} = 2a_0 \mathrm{Re}\left(\frac{a_1}{a_0}\right) = -2a_0 \mathrm{Re}\left(\sum_{\nu=1}^{n} z_\nu\right) \neq 0$$

(in view of the fact that we have

$$-\mathrm{Re}\left(\frac{a_1}{a_0}\right) = \mathrm{Re}\left(\sum_{\nu=1}^{n} z_\nu\right) < 0).$$

Let us put

$$\phi(z) = \frac{\bar{a}_0 f(z)}{a_0 f^*(z)}$$

and consider the function

$$\psi(z) = \frac{\phi(z)+1}{\phi(z)-1} = \frac{\bar{a}_0 f(z) + a_0 f^*(z)}{\bar{a}_0 f(z) - a_0 f^*(z)}. \tag{5.28}$$

As the function $v = (u-1)/(u+1)$ maps the circle $|u| < 1$ onto the half-plane $\mathrm{Re}\, v < 0$, it follows from the relations (5.17), (5.18) and (5.19), that

$$\left.\begin{array}{llll} \text{for} & \mathrm{Re}\,z < 0 & |\phi(z)| < 1, & \mathrm{Re}\,\psi(z) < 0, \\ \text{for} & \mathrm{Re}\,z = 0 & |\phi(z)| = 1, & \mathrm{Re}\,\psi(z) = 0, \\ \text{for} & \mathrm{Re}\,z > 0 & |\phi(z)| > 1, & \mathrm{Re}\,\psi(z) > 0. \end{array}\right\} \tag{5.29}$$

† In this connexion, let us remember, that we always have $1/q = (\bar{q}/|q|^2)$; hence the signs of $\mathrm{Re}(1/q)$ and $\mathrm{Re}\, q$ always coincide with one another.

Let us expand the rational function $\psi(z)$ into simple fractions. Its denominator, as we know, has $n-1$ simple roots $i\gamma_1, \ldots, i\gamma_{n-1}$ (where the γ_ν are real numbers). These points $i\nu_\nu$ are simple poles of the function $\psi(z)$.

In the numerator of the function $\psi(z)$ there stands a polynomial of the n-th degree; hence its integral part reduces to a linear function. As a simple calculation shows, it has the form

$$\frac{2a_0\bar{a}_0}{\bar{a}_0a_1+a_0\bar{a}_1}z+\text{const.}$$

Therefore, we find that

$$\psi(z) = \frac{2a_0\bar{a}_0}{\bar{a}_0a_1+a_0\bar{a}_1}z+\sum_{\nu=1}^{n-1}\frac{\lambda_\nu}{z-i\beta_\nu}+C, \tag{5.30}$$

where C is a constant.

It follows from the relation (5.29), that the sign of $\operatorname{Re}\psi(z)$ is identical with the sign of $\operatorname{Re}(z-i\beta_\nu)$ (the sign of the latter quantity does not depend on the subscript ν). Close to the point $z=i\beta_\nu$ the expression $\lambda_\nu/(z-i\beta_\nu)$ forms the *principal* part of the function $\psi(z)$; all the remaining terms of the expression (5.30) have no significant effect on its magnitude. Hence it follows, that the residues λ_ν are positive real numbers.†

Taking this fact into account, on calculating the quantity $\operatorname{Re}\psi(0)$, we also find that in formula (5.30) $\operatorname{Re} C = 0$. The circumstances indicated permit us to conclude, that the sign of the real part of the quantity

$$\psi_1(z) = \psi(z)-\frac{2a_0\bar{a}_0}{\bar{a}_0a_1+a_0\bar{a}_1}z = \sum_{\nu=1}^{n-1}\frac{\lambda_\nu}{z-i\beta_\nu}+C \tag{5.31}$$

is identical with the sign of $\operatorname{Re}\psi(z)$. Hence the function

$$\phi_1(z) = \frac{\psi_1(z)+1}{\psi_1(z)-1},$$

connected with the function $\psi_1(z)$ in the same way as the function

† If this were not the case the part of the neighbourhood of the point $z=i\beta_\nu$, where $\operatorname{Re}(z-i\beta_\nu) > 0$ could not be identical with the part of this neighbourhood, where $\operatorname{Re}[\lambda_\nu/(z-i\beta_\nu)] > 0$.

$\phi(z)$ is connected with the function $\psi(z)$, will satisfy the relations (5.29). We find that

$$\left.\begin{array}{ll} \text{for} \quad \operatorname{Re} z < 0 & |\phi_1(z)| < 1, \\ \text{for} \quad \operatorname{Re} z = 0 & |\phi_1(z)| = 1, \\ \text{for} \quad \operatorname{Re} z > 0 & |\phi_1(z)| > 1. \end{array}\right\} \tag{5.32}$$

Let us consider the function

$$f_1(z) = (a_0\bar{a}_1 + a_1\bar{a}_0 - a_0\bar{a}_0 z)f(z) + a_0^2 z f^*(z). \tag{5.33}$$

It is easy to see, that

$$\frac{f_1(z)}{f_1^*(z)} = \frac{a_0}{\bar{a}_0}\phi_1(z),$$

where $f_1^*(z)$ is the polynomial defined by formula (5.16), if $f_1(z)$ is substituted for the polynomial $f_1(z)$.

We shall also establish, that $f_1(z)$ is a polynomial of the $(n-1)$-th degree. A simple calculation again shows us that the coefficients of z^{n+1} and z^n in the expression (5.33) are equal to zero. On the other hand, from the last equation after replacing in it the function $\phi_1(z)$ by its expression in terms of $\psi(z)$, and then substituting the expression for $\psi_1(z)$ from formula (5.31) it is obvious, that the degree of $f_1(z)$ cannot be less than $n-1$. Hence it follows, that the polynomials $f_1(z)$ and $f_1^*(z)$ do not have common roots, the zeros of the functions $f_1(z)$ and $\phi_1(z)$ are identical with one another.

By virtue of the relations (5.35) $|\phi_1(z)| \geqslant 1$ for $\operatorname{Re} z \geqslant 0$. Thus, all the zeros of the function $\phi_1(z)$, and consequently, also of the polynomial $f_1(z)$ lie in the left hand half-plane. We have shown that the *polynomial* $f_1(z)$ *belongs to the class* (H). Let us note also: in the process of proof we have established, that subject to our conditions $\operatorname{Re}(a_1/a_0) > 0$.

The converse statement is also true: if $\operatorname{Re} a_1/a_0 > 0$ and the polynomial $f_1(z)$ belongs to the class (H), then the polynomial $f(z)$ also belongs to the class (H).

This statement is proved in the same way as the preceding, the only difference being that here the starting point of the discussion is the function $f_1(z)$.

Proceeding in the way indicated, we again arrive at the result that the function $\psi_1(z)$ satisfies the conditions (5.29) (as we have established this above for the function $\psi(z)$), then we have to pass

to the function $\psi(z)$, using formula (5.31). Taking into consideration the fact that $\mathrm{Re}(a_1/a_0) > 0$, we find that the function $\psi(z)$, like the function $\psi_1(z)$, also satisfies the conditions (5.29), and the function $\phi(z) = [\psi(z)+1]/[\psi(z)-1]$ satisfies the conditions (5.32). Finally, we repeat with reference to the functions $\phi(z)$ and $f(z)$ the reasoning, with the help of which it was shown, starting from the similar properties of the function $\phi_1(z)$, that the function $f_1(z)$ belonged to the class (H). In the present case this reasoning enables us to establish that the function $f(z)$ belongs to the class (H). The last result completes the proof of the following proposition:

THEOREM 3. *The polynomial $f(z)$ of the n-th degree belongs to the Hurwitz class (H), if and only if the polynomial of the $(n-1)$-th degree $f_1(z)$, defined by formula* (5.33) *is a member of this class, and* $\mathrm{Re}(a_1/a_0) > 0$.

Each of the two theorems, which we have proved, enables us with the help of not more than $n-1$ steps, to establish whether or not a given polynomial belongs to the Hurwitz class (H). The reader is recommended as an exercise to derive independently by means of the application of theorem 3, the results obtained for the quadratic equation in example 4 of the preceding article.

Let us now suppose, that all the coefficients a_ν of the polynomial $f(z)$ are real numbers. In this case it is convenient to put $f(z) = p(z)+$ $+q(z)$, where

$$p(z) = a_0 z^n + a_2 z^{n-2} + \dots,$$
$$q(z) = a_1 z^{n-1} + a_3 z^{n-3} + \dots;$$

then

$$f^*(z) = p(z) - q(z),$$
$$f_1(z) = (2a_0a_1 - a_0^2 z)[p(z)+q(z)] + a_0^2 z[p(z)-q(z)] =$$
$$= 2a_0[a_1 p(z) + (a_1 - a_0 z)q(z)].$$

In the formulation of theorem 3 it is obvious that the polynomial $f_1(z)$ can be replaced by the polynomial

$$F_1(z) = \frac{f_1(z)}{2a_0} = a_1 p(z) + (a_1 - a_0 z)q(z) =$$
$$= a_1^2 z^{n-1} + (a_1a_2 - a_0a_3)z^{n-2} +$$
$$+ a_1a_3 z^{n-3} + (a_1a_4 - a_0a_5)z^{n-4} + \dots$$
$$\dots + a_1a_{2m-1}z^{n-(2m-1)} + (a_1a_{2m} - a_0a_{2m+1})z^{n-2m} + \dots \tag{5.34}$$

Our aim is the proof of the following theorem.

THEOREM 4. (HURWITZ'S CRITERION). *The polynomial*

$$f(z) = a_0 z^n + \dots + a_n,$$

where $a_0 > 0$ and all the numbers $a_\nu (\nu = 1, \dots, n)$ are real, belongs to the Hurwitz class (H) if and only if

$$D_1 = a_1 > 0, \quad D_2 = \begin{vmatrix} a_1 & a_0 \\ a_3 & a_2 \end{vmatrix} > 0, \quad D_3 = \begin{vmatrix} a_1 & a_0 & 0 \\ a_3 & a_2 & a_1 \\ a_5 & a_4 & a_3 \end{vmatrix} > 0, \dots$$

$$\dots, D_m = \begin{vmatrix} a_1 & a_0 & 0 & \dots 0 \\ a_3 & a_2 & a_1 & \dots 0 \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ a_{2m-1} & a_{2m-2} & a_{2m-3} & \dots a_m \end{vmatrix} > 0, \dots$$

$$\dots, D_n = \begin{vmatrix} a_1 & a_0 & 0 & \dots 0 \\ a_3 & a_2 & a_1 & \dots 0 \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ a_{2n-1} & a_{2n-2} & a_{2n-3} & \dots a_n \end{vmatrix} > 0 \qquad (5.35)$$

(it is here assumed that $a_k = 0$ for $k > n$).

Proof. Let us form the determinant (5.35) for the polynomial $F_1(z)$:

$$\Delta_1 = a_1 a_2 - a_3 a_0, \qquad \Delta_2 = \begin{vmatrix} a_1 a_2 - a_0 a_3 & a_1^2 \\ a_1 a_4 - a_0 a_5 & a_1 a_3 \end{vmatrix}, \dots$$

It is easy to see, that

$$a_0 a_1 \Delta_1 = \begin{vmatrix} a_0 a_1 & 0 \\ a_0 a_3 & a_1 a_2 - a_3 a_0 \end{vmatrix} = \begin{vmatrix} a_0 a_1 & a_0 a_1 \\ a_0 a_3 & a_1 a_2 \end{vmatrix} = a_0 a_1 \begin{vmatrix} a_1 & a_0 \\ a_3 & a_2 \end{vmatrix} = a_0 a_1 D_1,$$

$$a_0a_1\Delta_2 = \begin{vmatrix} a_0a_1 & 0 & 0 \\ a_0a_3 & a_1a_2-a_3a_0 & a_1^2 \\ a_0a_5 & a_1a_4-a_0a_5 & a_1a_3 \end{vmatrix} =$$

$$= \begin{vmatrix} a_0a_1 & a_0a_1 & 0 \\ a_0a_3 & a_1a_2 & a_1^2 \\ a_0a_5 & a_1a_4 & a_1a_3 \end{vmatrix} = a_0a_1^2 \begin{vmatrix} a_1 & a_0 & 0 \\ a_3 & a_2 & a_1 \\ a_5 & a_4 & a_3 \end{vmatrix} = a_0a_1^2D_3$$

(we add to the elements of the second columns the elements of the first ones and take out of the determinant the common factor which is obtained in the columns):

$$a_0a_1\Delta_3 = \begin{vmatrix} a_0a_1 & 0 & 0 & 0 \\ a_0a_3 & a_1a_2-a_0a_3 & a_1^2 & 0 \\ a_0a_5 & a_1a_4-a_0a_5 & a_1a_3 & a_1a_2-a_0a_3 \\ a_0a_7 & a_1a_6-a_0a_7 & a_1a_5 & a_1a_4-a_0a_5 \end{vmatrix} =$$

$$= \begin{vmatrix} a_0a_1 & a_0a_1 & 0 & 0 \\ a_0a_3 & a_1a_2 & a_1^2 & a_0a_1 \\ a_0a_5 & a_1a_4 & a_1a_3 & a_1a_2 \\ a_0a_7 & a_1a_6 & a_1a_5 & a_1a_4 \end{vmatrix} = a_0a_1^3 \begin{vmatrix} a_0 & a_1 & 0 & 0 \\ a_3 & a_2 & a_1 & a_0 \\ a_5 & a_4 & a_3 & a_2 \\ a_7 & a_6 & a_5 & a_4 \end{vmatrix} = a_0a_1^3D_4$$

(we add to the elements of the second column the elements of the first, to the elements of the fourth column the elements of the third, multiplied by a_1/a_0, and then take out of the determinant the common factor which is obtained in the columns). We obtain the similar relations between $\Delta_{\nu-1}$ and D_ν for all ν for $4 < \nu < n$, in a similar way.

Let us now turn directly to the proof of theorem 4. To begin with let us note, that it is true for $n = 1$. In this case the conditions (5.35) reduce to the single one: $a_1 > 0$. It is obvious, that for $a_0 > 0$ and $a_1 > 0$ the equation $f(z) = a_0+a_1z = 0$ has a negative real root. Hence it is sufficient to show, that theorem 4 is true for polynomials of the n-th degree, when it has been assumed that it is true for polynomials of the $(n-1)$-th degree. Later on we shall start from this assumption.

In virtue of theorem 3 the polynomial $f(z)$ belongs to the class (H) if and only if $a_1 > 0$, and the polynomial $F_1(z)$ belongs to this class. By our assumption theorem 4 is true for the polynomial

$F_1(z)$. For it to belong to the class (H) (and consequently for $a_1 > 0$ also the polynomial $f(z)$), it is necessary and sufficient that the inequalities

$$\Delta_1 > 0, \quad \Delta_2 > 0, \quad \ldots, \quad \Delta_{n-1} > 0$$

should be satisfied. Thanks to the relations between the determinants $\Delta_{\nu-1}$ and D_ν obtained above, these inequalities are (for $a_1 > 0$) equivalent to the conditions

$$D_2 > 0, \quad D_3 > 0, \quad \ldots, \quad D_n > 0.$$

Therefore, we arrive at the result, that for the polynomial $f(z)$ to belong to the class (H) it is necessary and sufficient, that the inequalities

$$D_1 = a_1 > 0, \quad D_2 > 0, \quad \ldots, \quad D_n > 0$$

should hold.

Example 1. Let it be required to establish whether or not the polynomial

$$f(z) = z^5 + z^4 + 5z^3 + 7z^2 + 4z + 8$$

belongs to the class (H). We will apply theorem 2 to the solution of this problem; let us take $\zeta = -1$. In our case

$$f^*(z) = z^5 - z^4 + 5z^3 - 7z^2 + 4z - 8,$$

$$f(-1) = 6, \quad f^*(1) = -26,$$

$$f^*(-1)f(z) - f(-1)f^*(z) = -4(8z^5 + 5z^4 + 40z^3 + 35z^2 + 32z + 40),$$

$$f_1(z) = -4(8z^4 - 3z^3 + 43z^2 - 8z + 40).$$

It is obvious, that the sum of the roots of the polynomial $f_1(z)$ is equal to $\frac{3}{8}$. This could not be so if the real parts of all the roots were negative. Thus, we arrive at the result, that the polynomial $f_1(z)$, and consequently, also the polynomial $f(z)$, does not belong to the Hurwitz class (H).

Example 2. Let us investigate whether the polynomial

$$f(z) = z^4 + 2(1+i)z^3 - (2-2i)z^2 - 8(1+i)z - 4(1+2i)$$

belongs to the Hurwitz class. We will base the investigation on theorem 3. In the first place let us establish, that in our case

$$\mathrm{Re}\left(\frac{a_1}{a_0}\right) = \mathrm{Re}[2(1+i)] = 2 > 0.$$

We then have to find the polynomial $f_1(z)$. Using formulas (5.16) and (5.32), we obtain

$$f^*(z) = z^4-2(1-i)z^3-(3+2i)z^2-8(1-i)z-4(1-2i),$$
$$f_1(z) = \tfrac{4}{5}(2+i)[5z^3+(4+3i)z^2-20z-4(4+3i)].$$

Let us now apply theorem 3 to the polynomial $f_1(z)$ (in this we neglect the factor $4/5(2+i)$, which stands in front of the square brackets and does not influence the distribution of its zeros). Moreover, we will denote the coefficients of this polynomial by $a_\nu^{(1)}$.

In the first place we find, that

$$\mathrm{Re}\frac{a_1^{(1)}}{a_0^{(1)}} = \mathrm{Re}\frac{4+3i}{5} = \frac{4}{5} > 0.$$

Then by formula (5.33) we form the polynomial $f_2(z)$, which stands in the same relation to the polynomial $f_1(z)$ as the polynomial $f_1(z)$ itself does to the polynomial $f(z)$. We have

$$f_1^*(z) = 5z^3-(4-3i)z^2-20z+4(4-3i),$$
$$f_2(z) = 40(4+3i)(z^2-4).$$

Denoting the coefficients of the last polynomial by $a_\nu^{(2)}$, we find, that

$$\mathrm{Re}\left(\frac{a_1^{(2)}}{a_0^{(2)}}\right) = 0.$$

Thus, we arrive at the result, that the polynomial $f(z)$ does not belong to the class (H), and in consequence, neither does $f_1(z)$ or the given polynomial $f(z)$.

Example 3. Let us investigate whether the polynomial

$$f(z) = z^4+4z^3+9z^2+8z+5$$

belongs to the class (H).

We make use of Hurwitz's criterion (theorem 4). We have

$$D_0 = a_0 = 1, \quad D_2 = \begin{vmatrix} a_1 & a_0 \\ a_3 & a_2 \end{vmatrix} = \begin{vmatrix} 4 & 1 \\ 8 & 9 \end{vmatrix} = 28 > 0,$$

$$D_3 = \begin{vmatrix} a_1 & a_0 & 0 \\ a_3 & a_2 & a_1 \\ a_5 & a_4 & a_3 \end{vmatrix} = \begin{vmatrix} 4 & 1 & 0 \\ 8 & 9 & 4 \\ 0 & 5 & 8 \end{vmatrix} = 144 > 0,$$

$$D_4 = \begin{vmatrix} a_1 & a_0 & 0 & 0 \\ a_3 & a_2 & a_1 & a_0 \\ a_5 & a_4 & a_3 & a_2 \\ a_7 & a_6 & a_5 & a_4 \end{vmatrix} = \begin{vmatrix} 4 & 1 & 0 & 0 \\ 8 & 9 & 4 & 1 \\ 0 & 5 & 8 & 9 \\ 0 & 0 & 0 & 5 \end{vmatrix} = 5 \begin{vmatrix} 4 & 1 & 0 \\ 8 & 9 & 4 \\ 0 & 5 & 8 \end{vmatrix} > 0.$$

Therefore, all the determinants D_ν are positive. Consequently, $f(z)$ is a Hurwitz polynomial.

29. Other methods

Besides the theorem, proved in Art. 27, there exist many other propositions, which indicate the necessary and sufficient conditions for some polynomial to belong to the Hurwitz class. We intend now to give some of them (without proof).

To begin with we will pause at one theorem, due to I. Schur, which is closely connected in its character with the theory developed in the preceding article.

THEOREM 5. *Let ζ be a certain fixed number with negative real part. The polynomial $f(z)$ of the n-th degree will belong to the Hurwitz class (H), if and only if* (1) $\text{Re}\,(a_1/a_0) > 0$ *and* (2) *the polynomial*

$$H(z) = z^{-2}[(\bar{a}_0 z - a_1 \zeta z + \bar{a}_0 \zeta) f(z) - (a_0 z - a_1 z \zeta + a_0 \zeta) f^*(z)]$$

of the $(n-1)$-th degree belongs to the Hurwitz class (H).†

The problem of finding conditions, with the satisfaction of which all the roots of a certain polynomial will lie in the left hand half-plane was proposed to Hurwitz by his colleague at the Zürich Polytechnic the professor of mechanics A. Stodola. Stodola himself also attempted to solve this problem and derived a criterion, which asserted, that in order that a certain polynomial with real coefficients should belong to the class (H) it is necessary and sufficient, that *all its coefficients should have one and the same sign.*

In fact, the condition indicated is not sufficient, as is seen even from example 1 of the preceding article. All the coefficients of the polynomial $f(z) = z^5+z^4+5z^3+7z^2+4z+8$ considered there, are positive, however, it is not a Hurwitz polynomial.

At the same time, the *satisfaction of the conditions of Stodola is necessary for a polynomial with real coefficients to belong to the class (H).* In fact, the roots of the polynomial $f(z)$ with real coefficients

† The proof of this theorem can, for example, be found in the text book: A. G. Kurosh, *Course of Higher Algebra* (Gostekhizdat, 1959), page 284.

are either real numbers, or pairs of conjugate complex numbers. Let p_μ be the roots of the first kind, $-p_\nu \pm iq_\nu$ be the roots of the second kind. Then

$$P(z) = a_0 \prod_\mu (z+p_\mu) \prod_\nu [(z+p_\nu)^2+q_\nu^2].$$

If all the $p_\mu > 0$ and all the $p_\nu > 0$, then the signs of all the coefficients of the polynomial $f(z)$ are identical with the sign of the coefficient a_0.

The question of whether the polynomial $f(z)$ belongs to the class of Hurwitz (H) can also be solved with the aid of a certain series of Sturm, connected with this polynomial. We will give the rule relating to this in the form, which is due to H. G. Chebotarev.†

Let us start with the polynomial $F(z) = f(iz)$. Let $c_\nu = \alpha_\nu + i\beta_\nu$ be its coefficients. We shall suppose, that the coefficient c_0 of z^n in the polynomial $F(z)$ is a real number different from zero (that is $\alpha_0 \neq 0$, $\beta_0 = 0$). If necessary this can always be attained by multiplying the polynomial $f(z)$ by a suitable constant multiplier (an operation, which does not change the distribution of the roots of the given polynomial).

Let us consider the polynomials with real coefficients

$$r(z) = \alpha_0 z^n + \alpha_1 z^{n-1} + \dots + \alpha_n,$$

$$s(z) = \beta_1 z^{n-1} + \dots + \beta_n.$$

Let us form the Sturm series for the polynomials $r(z)$ and $s(z)$. Let us put $V(z) = r(z)$, $V_1(z) = s(z)$, and let us divide the polynomial $V(z)$ by the polynomial $V_1(z)$. Let us denote by $V_2(z)$ the remainder from this division taken with negative sign. Therefore, we have $V(z) = V_1(z)d_1(z) - V_2(z)$, where $d_1(z)$ is the polynomial, obtained as quotient.

Then let us divide the polynomial $V_1(z)$ by the polynomial $V_2(z)$. Let $V_3(z)$ be the remainder from this division taken with negative sign (therefore, we have $V_1(z) = V_2(z)d_2(z) - V_3(z)$, where $d_2(z)$ is the polynomial, obtained as quotient). The next step will consist in finding a polynomial $V_4(z)$, equal to the remainder taken

† See H. G. Chebotarev and I. I. Meiman, *The Routh–Hurwitz Problem for Polynomials and Integral Functions* (Trudy matem. in-ta Akad. Nauk SSSR, No. XXVI, 1949), pages 26–28.

with negative sign from the division of the polynomial $V_3(z)$ by the polynomial $V_2(z)$, and so on.

We shall proceed in this way, until the next polynomial $V_{m-1}(z)$ is divisible by the polynomial $V_m(z)$ which follows it in our sequence, without remainder. Then the polynomials

$$V(z), \quad V_1(z), \quad \ldots, \quad V_m(z) \tag{5.36}$$

form the Sturm series which we require. Here $V_m(z)$ is the greatest common divisor of the polynomial $r(z)$ and $s(z)$. If these polynomials are prime to one another, the polynomial $V_m(z)$ reduces to a constant.

The question of whether the polynomial $f(z)$ belongs to the class (H) is solved, if $V_m =$ constant, as a result of theorem 6; if the degree of V_m is greater than zero, it is solved as a result of theorem 7. We will give these theorems without proof.

THEOREM 6. *If the degrees of any two adjacent polynomials of the series* (5.36) *differ from one another by unity, the coefficients of their highest degrees always have different signs and the last term of the series* (5.36) *is* $V_m =$ constant, *then the polynomial* $f(z)$ *belongs to the Hurwitz class* (H).

If the coefficients of the terms of highest degree of even one pair of adjacent polynomials of the series (5.36) *have the same sign, then* $f(z)$ *is neither an H-polynomial nor an H'-polynomial.*

Let us now assume, that the degree of the polynomial $V_m(z)$ is higher than zero, but the remaining conditions, which by virtue of theorem 6 ensure that the polynomial $f(z)$ belongs to class (H), are satisfied. We shall construct in addition to the series (5.36) another Sturm series, formed from the polynomial $V_m(z)$ and its derivative $V'_m(z)$. Let us put $V_m(z) = U(z)$, $V_m'(z) = U_1(z)$ and for $U_2(z)$ let us take the remainder with negative sign obtained from the division of the polynomial $U(z)$ by the polynomial $U_1(z)$, for $U_3(z)$ let us take the remainder with negative sign from the division of the polynomial $U_1(z)$ by $U_2(z)$ and so on. As a result we obtain the Sturm series

$$U(z), \quad U_1(z), \quad \ldots, \quad U_p(z). \tag{5.37}$$

With its help we can formulate

THEOREM 7. *Let the degree of the polynomial* $V_m(z)$ *from the series* (5.36) *be higher than zero, but the remaining conditions, which, by*

virtue of theorem 6, ensure that the polynomial $f(z)$ belongs to the class (H), are satisfied.

Then, if the degrees of any two adjacent polynomials of the series (5.37) always differ from one another by unity, and all the coefficients of their terms of highest degree have the same signs, $f(z)$ is an H'-polynomial (but not an H-polynomial). If not all the indicated coefficients have identical signs, then $f(z)$ is neither an H-polynomial nor an H'-polynomial.

Let us note also, that the properties of the series (5.36) and (5.37) considered in theorems 6 and 7 are not changed on multiplying the polynomials $V_k(z)$ and $U_k(z)$ by constant positive multipliers. This circumstance at times permits us to simplify the calculations, carried out in order to construct the series (5.36) and (5.37).

Example 1. Let us investigate whether the polynomial

$$f(z) = z^4+7z^3+21z^2+31z+20$$

belongs to the class (H) or (H'). We find successively, that

$$F(z) = f(iz) = z^4-7iz^3-21z^2+31iz+20,$$
$$r(z) = V(z) = z^4-21z^2+20;$$
$$s(z) = V_1(z) = -7z^3+31z;$$
$$V_2(z) = 29z^2-35, \quad V_3(z) = -z, \quad V_4(z) = 1.$$

First we divide the polynomial $V(z)$ (multiplied by 7) by $V_1(z)$. We take the remainder obtained with negative sign and neglect the factor 4; we take the result as $V_2(z)$. Then we divide the polynomial $V_1(z)$ (multiplied by 29) by $V_2(z)$. We take the remainder obtained with negative sign and neglect the factor 654; we take the result as $V_3(z)$. Finally, we divide the polynomial $V_2(z)$ by $V_3(z)$; change the sign of the remainder and neglect the factor 35. As a result we obtain, that $V_4(z) = 1$.

The Sturm series which consists of the polynomials V, V_1, V_2, V_3, V_4, possesses all the properties, indicated in theorem 6 as sufficient conditions for the polynomial $f(z)$ to belong to the class (H). Therefore, we arrive at the result, that $f(z)$ is a Hurwitz polynomial.

Example 2. Let us investigate whether the polynomial

$$f(z) = z^4+(3+2i)z^3+(4+5i)z^2+(1+8i)z-3(1-i)$$

belongs to the class (H) or to the class (H'). We find in succession, that

$$F(z) = f(iz) = z^4+(2-3i)z^3+(-4-5i)z^2+(-8+i)z+(-3+3i),$$

$$r(z) = V(z) = z^4+2z^3-4z^2-8z-3,$$

$$s(z) = V_1(z) = -3z \ -5z^2+z+3,$$

$$V_2(z) = 19z^2+31z+12,$$

$$V_3(z) = -z-1.$$

Here $V_2(z)$ (taken with negative sign, multiplied by $\frac{3}{2}$) is the remainder from the division of the polynomial $V(z)$ (multiplied by 3) by the polynomial $V_1(z)$. The polynomial $V_3(z)$ is the remainder (taken with negative sign; the factor 1107 neglected) from the division of the polynomial $V_1(z)$ (multiplied by 19) by the polynomial $V_2(z)$. The polynomial $V_2(z)$ is divisible without remainder by the polynomial $V_3(z)$.

The Sturm series, which consists of the polynomials V, V_1, V_2, V_3, finishes with the polynomial $V_3(z)$, different from a constant. On the other hand, it possesses all the remaining properties, indicated in theorem 6 as sufficient conditions for the polynomial $f(z)$ to belong to the class (H). Hence for the final solution of the given problem it is necessary to construct the Sturm series for the polynomials $U(z) = V_3(z) = -z-1$ and $U_1(z) = V'_3(z) = -1$. However the polynomials $U(z)$ and $U_1(z)$ themselves form the given Sturm series (insofar as $U_1(z)$ = constant). This series possesses all the properties, indicated in theorem 7 as sufficient conditions for the polynomial $f(z)$ to belong to the class (H').

Therefore, we arrive at the result, that $f(z)$ is an H'-polynomial.

We shall not stop for other criteria for polynomials with constant coefficients to belong to the classes (H) and (H')† (the case of polynomials with coefficients, which depend on parameters, is considered in the next article).

In many technical problems Hurwitz's problem has to be solved for functions of a more general type, than the polynomial. As such functions the functions known as "quasi polynomials" are often

† Besides the literature indicated in the two preceding remarks, see the book: M. G. Krein and M. A. Neimark, *The Method of Symmetrical and Hermite Forms in the Theory of the Separation of the Roots of Algebraic Equations* (GNTI Ukraine, 1936).

encountered. Functions of the form

$$f(z) = p_1(z)e^{\lambda_1 z} + \dots + p_m(z)e^{\lambda_m z},$$

are usually called quasi polynomials where $\lambda_1, \dots \lambda_m$ are real numbers, $p_1(z), \dots, p_m(z)$ are polynomials. It is said, that a certain function (in particular, a quasi polynomial) is an H-function or, otherwise, belongs to the Hurwitz class (H), if the real parts of all its roots are negative.

Criteria, which enable us in many cases to decide the question of whether a given quasi polynomial belongs to the class (H), were found by L. S. Pontriagin. More general results were then obtained by H. G. Chebotarev. Problems, connected with the distribution of quasi polynomials of particular types with coefficients, which depend on parameters (a circumstance, which is extremely material from the point of view of applications), were subsequently considered by pupils of H. G. Chebotarev (V. I. Tsapirin and others).

The case of integral functions† has been considered by many authors. It is possible to familiarize oneself with the results obtained in this field (as also with the results relating to quasi polynomials) in the monograph by H. G. Chebotarev and N. N. Meiman already cited.

30. Polynomials depending on parameters. The Vishnegradskii-Nyquist method

In Art. 26 we have already (in the consideration of simple examples) touched upon the case, where the coefficients of the polynomial $f(z) = a_0z^n + a_1z^{n-1} + \dots + a_{n-1}z + a_n$ depend on parameters. Let us now consider this important case in greater detail, limiting ourselves to the following statement of the problem. Let the coefficients of the polynomial $f(z)$ depend linearly on two parameters ξ and η, which we shall treat as rectangular co-ordinates on some auxiliary plane. Thus, to each point of this plane there corresponds a definite polynomial, and conversely. *Find in the ξ, η-plane that domain (the domain of stability), which consists of points (ξ, η), which correspond to H-polynomials $f(z)$.*

Let us begin with some general observations. The continuous dependence of the coefficients of the polynomial $f(z)$ on the parameters ξ, η entails the continuous dependence of the roots of these

† It is obvious that quasi polynomials are integral functions. For the definition of an integral function see F.C.V., Chap. VI, Art 71.

polynomials on ξ and η at all the points (ξ, η), apart from those, at which the coefficient a_0 of the highest term becomes zero, as at these points, at least, one root of $f(z)$ becomes infinite. Let us denote by $D(k, n-k)$ the set of points of the ξ, η-plane, which corresponds to the polynomials $f(z)$, which have k roots with negative real part and $n-k$ roots with positive real part $(k = 0, 1, \ldots, n)$. If $D(k, n-k)$ is not an empty set (which for certain k is possible) then, as it is not difficult to see, it will consist only of interior points, that is together with each point which belongs to it, it will also contain a sufficiently small circle with centre at this point. Also the domains $D(k, n-k)$ will occupy the whole of the ξ, η-plane and on the boundary of each of them, at least one of the roots of $f(z)$ must be imaginary (have real part equal to zero) or infinite.

Every polynomial, the coefficients of which depend linearly on the two parameters ξ and η, can be written in the form

$$f(z) = \xi P(z)+\eta Q(z)+R(z),$$

where $P(z)$, $Q(z)$ and $K(z)$ are polynomials with constant coefficients of degrees, not exceeding n, and, at least one of these three polynomials must be of degree n. We shall search for the geometrical locus (N) of points (ξ, η), for which $f(z)$ has, at least one imaginary finite root. For this let us put $z = iy$ and

$$P(iy) = P_1(y)+iP_2(y), \quad Q(iy) = Q_1(y)+iQ_2(y),$$

$$R(iy) = R_1(y)+iR_2(y),$$

where $P_1(y)$, $P_2(y)$, $Q_1(y)$, $Q_2(y)$, $R_1(y)$ and $R_2(y)$ are polynomials with real coefficients. In this notation

$$f(z) = \xi P_1(y)+\eta Q_1(y)+R_1(y)+i\{\xi P_2(y)+\eta Q_2(y)+R_2(y)\},$$

and any real root y of this polynomial must simultaneously satisfy the two equations:

$$\xi P_1(y)+\eta Q_1(y)+R_1(y) = 0, \qquad (5.38_1)$$

$$\xi P_2(y)+\eta Q_2(y)+R_2(y) = 0. \qquad (5.38_2)$$

Conversely, if ξ and η are such, that equations (5.38_1) and (5.38_2) have a common real root y, then iy is a root of $f(z)$ and the point (ξ, η) belongs to (N).

Thus, if $P_1(y), Q_2(y) - P_2(y) Q_1(y) \not\equiv 0$, then the required geometrical locus (N) has the following parametric equations:

$$\begin{aligned} \xi &= \frac{R_1(y)\,Q_2(y) - R_2(y)\,Q_1(y)}{\Delta(y)}, \\ \eta &= \frac{P_1(y) R_2(y) - P_2(y) R_1(y)}{\Delta(y)}, \end{aligned} \tag{5.39}$$

where $\Delta(y) = P_1(y) Q_2(y) - P_2(y) Q_1(y)$, and the parameter y runs through all real values from $-\infty$ to ∞. The geometrical locus (5.39) together with the straight line $a_0 = 0$ divides the ξ, η-plane into several domains, which are the domains $D(k, n-k)$. In order to establish what value of k corresponds to each of these domains, it is possible to choose a point (ξ, η) within the domain considered, for which the character of the roots of the polynomial $f(z)$ is most simply established, and determine the number of roots of the polynomial with negative real part, which correspond to this point. This number will also, obviously, be the number k for the whole of the domain. Let us consider a simple example.

Example 1. *The domain of stability for the polynomial*

$$f(z) = z^3 + \eta z^2 + \eta z + \xi.$$

Putting $z = iy$, we find

$$f(iy) = \xi - \eta y^2 + i(\eta y - y^3),$$

that is in the given case $P_1(y) = 1$, $Q_1(y) = -y^2$, $R_1(y) = 0$, $P_2(y) = 0$, $Q_2(y) = y$, $R_2(y) = -y^3$. Consequently, we must put

$$\xi - \eta y^2 = 0, \quad \eta y - y^3 = 0,$$

whence *either*

$$\xi = y^4, \quad \eta = y^2,$$

or (for $y = 0$)

$$\xi = 0, \quad \eta \quad \text{arbitrary}.$$

In the first case $\xi = \eta^2$, but η assumes only negative values, that is we have the half parabola $\xi = \eta^2$, $\eta \geqslant 0$ and in the second case we have the whole of the η-axis (Fig. 27). These then comprise in the given case the geometrical locus (N), which divides the ξ, η-plane, obviously, into three parts. The domain (I) contains the points for which $\eta = 0$, $\xi > 0$, which corresponds to the polynomial

$f(z) = z^3+\xi(\xi > 0)$, which has one real negative root and two roots with positive real part. Thus, the domain (I) is the domain D (1, 2). The domain (II) contains the point $\xi = 1$, $\eta = 3$, to which corresponds the polynomial $f(z) = z^3+3z^2+3z+1 = (z+1)^3$, which has all three roots in the left hand half-plane $\operatorname{Re} z < 0$. Consequently, the domain (II) is the domain D (3, 0), that is the domain of stability. The domain (III) contains the points, for which $\eta = 0$ and $\xi < 0$, corresponding to the polynomials $f(z) = z^3+\xi(\xi < 0)$, which have one real positive root and two roots with negative real part. Hence the domain (III) is the domain $D(1, 2)$. Let us note, that the domain D (0, 3) is empty, as the polynomial considered does not have three roots with positive real part even for a single set of values of ξ, η.

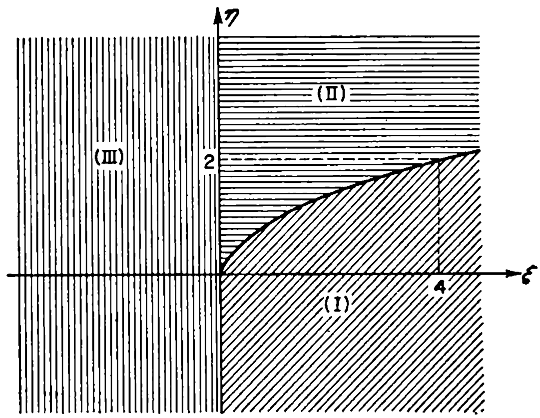


FIG. 27

If we put $\eta = 2$ (see example 3, Art. 26), the point $(\xi, 2)$ belongs, as is seen from Fig. 27, to the domain of stability $D(3, 0)$ for

$$0 < \xi < 4.$$

In the following example there appear certain typical peculiarities of the construction and investigation of the geometrical locus (N) and the domains $D(k, n-k)$.

Example 2. *The domain of stability for a polynomial of the third degree with arbitrary real coefficients.* If in the polynomial $f(z) = a_0z^3+a_1z^2+a_2z+a_3$ the coefficients a_0, a_1, a_2 and a_3 are real linear functions of ξ and η, where the straight lines $a_0 =$ constant and $a_1 =$ constant are not parallel, then it is possible to introduce new

parameters $a_0 = \Xi$ and $a_1 = \mathrm{H}$, on which ξ and η depend linearly. Thus, without restricting the generality of the treatment, it is possible to take (returning to the old notation),

$$f(z) = \xi z^3+\eta z^2+(a_2'+a_2''\xi+a_2'''\eta)z+a_3'+a_3''\xi+a_3'''\eta.$$

Putting $z = iy$ and separating the real and imaginary parts, we find, that

$$-\xi y^3+(a_2'+a_2''\xi+a_2'''\eta)y = 0,$$
$$-\eta y^2+a_3'+a_3''\xi+a_3'''\eta) = 0,$$

that is, *either*

$$\xi y^2 = a_2'+a_2''\xi+a_2'''\eta \text{ and } \eta y^2 = a_3'+a_3''\xi+a_3'''\eta,$$

or (for $y = 0$)

$$a_3'+a_3''\xi+a_3'''\eta = 0.$$

In the first case

$$\frac{a_2'+a_2''\xi+a_2'''\eta}{\xi} = \frac{a_3'+a_3''\xi+a_3'''\eta}{\eta},$$

or

$$a_3''\xi^2+(a_3'''-a_2'')\xi\eta-a_2'''\eta^2+a_3'\xi-a_2'\eta = 0, \qquad (5.40)$$

but

$$\frac{a_3'+a_3''\xi+a_3'''\eta}{\eta} \geqslant 0,$$

that is we have part of the curve of the second order (5.40), lying in that pair of angular sectors formed by the straight lines $\eta = 0$ and $a_3'+a_3''\xi+a_3'''\eta = 0$, in which the left hand sides of these equations have the same sign. In the second case we have the straight line $a_3'+a_3''\xi+a_3'''\eta = 0$. Hence, the geometrical locus (N) in the given example consists of the parts of the curve of the second order (5.40) indicated above and this straight line. In order to obtain the domains $D(k, n-k)$ it is however, necessary, to add to (N) the η-axis, along which the coefficient of the highest term vanishes. Let us carry out a detailed investigation in the following particular cases.

(*a*) The *polynomial* $\xi z^3+\eta z^2+(\xi+\eta)z-\xi+\eta+1$. Here the curve of the second order (5.40) has the equation $\xi^2+\eta^2-\xi = 0$, that is, it is a circle with centre at the point $(\frac{1}{2}, 0)$ and radius $\frac{1}{2}$ (Fig. 28a).

Three quarters of this circle belong to the geometrical locus (N), which lies above the straight line $\xi+\eta = 0$. The straight line $\xi+\eta+1 = 0$ also belongs to (N). It remains to establish what values of k correspond to each of the domains, into which (N) and the straight line $\xi + 0$ divide the ξ, η-plane.

Putting $\eta = 0$, we shall have the polynomial $\xi z^2+\xi z-\xi+1$ or $\xi(z^3+z^2-1+1/\xi)$. For $\xi < 0$ and $\xi > 1$ the absolute term $-1+1/\xi < 0$, and the equation $z^3+z-\lambda = 0$, where $\lambda > 0$, has one real positive root and two complex-conjugate roots with negative real part. Consequently, the domain, shaded on Fig. 28a parallel to the bisector of the first and third quadrants, is the domain $D(2, 1)$. For $0 < \xi < 1$ the absolute term $-1+1/\xi > 0$, and the equation $z^3+z+\lambda = 0$, where $\lambda > 0$, has one real negative root and two complex-conjugate roots with positive real part. Consequently, the domain, shaded parallel to the bisector of the second and fourth quadrants, is $D(1, 2)$. Putting $\xi = 1$, we shall have the polynomial $z^3+\eta z^2+(1+\eta)z+\eta$, which, for example, for $\eta = 2$ has the roots -1 and $-\frac{1}{2}\pm i(\sqrt{7}/2)$. Consequently, the upper horizontally shaded domain is the domain $D(3, 0)$. In exactly the same way, putting, for example, $\eta = -2$, $\xi = -\frac{1}{3}$, we shall have the polynomial $-\frac{1}{3}z^3-2z^2-\frac{7}{3}z-\frac{2}{3}$, which has the roots -1 and $-\frac{5}{2}\pm\sqrt{(17)}/2$, that is all three roots of which are real and negative. But the point $(-\frac{1}{3}, -2)$ belongs to the lower horizontally shaded domain, so that this latter also belongs to $D(3, 0)$. It remains to define the number k for the domain which has the form of a segment of the circle and is shaded on Fig. 28a vertically. Let us consider the point $(\frac{3}{4}, -\frac{23}{56})$, which belongs to this domain. To this point corresponds the polynomial

$$\tfrac{3}{4}z^3-\tfrac{23}{56}z^2+\tfrac{19}{56}z-\tfrac{9}{56},$$

which has roots $\frac{1}{2}$ and

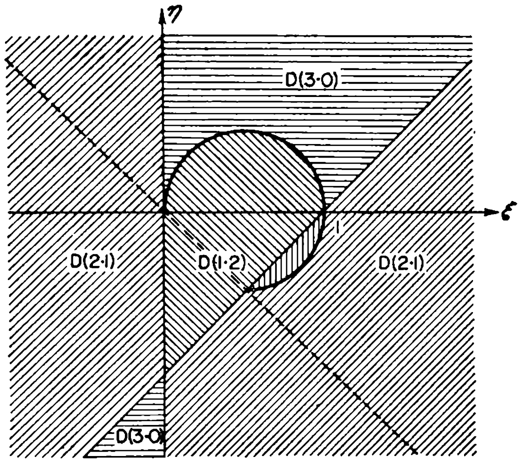
$$\tfrac{1}{42}\pm i\sqrt{(755)}/42.$$

Thus, this domain is the domain $D(0, 3)$.

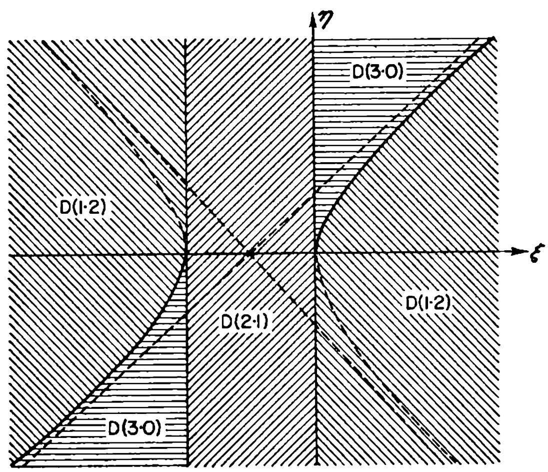
We note the fact, that in the concrete example considered the domain of stability (as also the domain $D(2, 1)$) consists of two parts.

(*b*) The *polynomial* $\xi z^3+\eta z^2+\eta z+\xi+1$. Here the curve (5.40) has the equation $\xi^2-\eta^2+\xi = 0$, that is, it is an equilateral hyperbola with centre at the point $(-\frac{1}{2}, 0)$, and asymptotes, parallel to the bisectors of the co-ordinate angles, and semi-axes, equal to $\frac{1}{2}$ (Fig. 28b). To the geometrical locus (N) belong: the upper half of the

right hand branch and the lower half of the left hand branch of the hyperbola. To (N) also belongs the straight line $\xi = -1$. It remains to establish what values of k correspond to each of the domains,



(a)



(b)

FIG. 28

into which (N) and the straight line $\xi = 0$ divide the ξ, η-plane. The reader will have no difficulty in convincing himself of the correctness of the data of Fig. 28b, by putting first $\eta = 0$ and investigating the polynomial for $\xi > 0$, $\xi < 1-$ and $-1 < \xi < 0$,

and then putting, for example, $\xi = 1$, $\eta = 3$ (the roots of the polynomial are -2 and $-\frac{1}{2} \pm i\sqrt{(3)}/2$ and $\xi = -2$, $\eta = -15/2$ (the roots of the polynomial are -2 and $-7/8 \pm \sqrt{(33)}/8$.

The basic idea of the given method is due to the Russian engineer I. A. Vishnegradskii, who developed in it detail in the year 1877 by an example of a polynomial of the third order. In the year 1932 this method was elaborated by Nyquist for polynomials of any degree in the special case, where $Q(z) = iP(z)$, that is, where $f(z) = P(z)(\xi + i\eta)$. In this case the geometrical locus (N) is said to be a Nyquist curve. With corresponding changes the Vishnegradskii–Nyquist method can also be applied to more general classes of

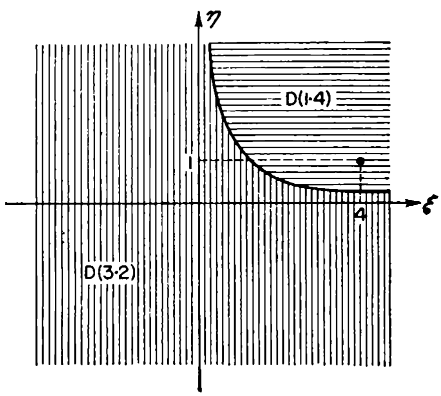


FIG. 29

functions. In particular, to quasi polynomials (see the end of Art. 28). The Nyquist curves of quasi polynomials were investigated in detail by Y. I. Naimark.

Let us note in conclusion, that the Vishnegradskii–Nyquist method can sometimes conveniently be applied even in the case of polynomials with constant coefficients by way of the artificial introduction of the parameters ξ and η. We will illustrate this remark by the example of the polynomial $z^5 + 4z^3 + z^2 + 1$. Let us consider in place of it the polynomial $z^5 + \xi z^3 + \eta z^2 + 1$, for which the geometrical locus (N) consists of the branch of the hyperbola $\xi\eta = 1$, which lies in the first quadrant (in fact, putting $z = iy$, we find the equations $-\eta y^2 + 1 = 0$ and $y^5 - \xi y^3 = 0$, and as $y \neq 0$ by virtue of the first of these equations, $y^2 = 1/\eta = \xi$). But for $\xi = 0$, $\eta = 0$ we obtain the polynomial $z^5 + 1$ with three roots in the left hand half-plane,

and consequently, the domain, which lies on the left and beneath (N), is $D(3, 2)$ (Fig. 29). Also for $\xi = \eta$ we obtain the polynomial

$$z^5+1+\xi z^2(z+1) = (z+1)\{z^4-z^3+(1+\xi)z^2-z+1\},$$

which has the root -1 and for $\xi = 5/4$ also the roots $\frac{1}{4} \pm i\sqrt{(15)}/2$, each of multiplicity 2. As the point (5/4, 5/4) lies above (N), the whole of the domain above (N) is the domain $D(1, 4)$. The given polynomial with constant coefficients corresponds to the point $\xi = 4$, $\eta = 1$, which belongs to the domain $D(1, 4)$. Consequently, this polynomial has only one root with negative real part.

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